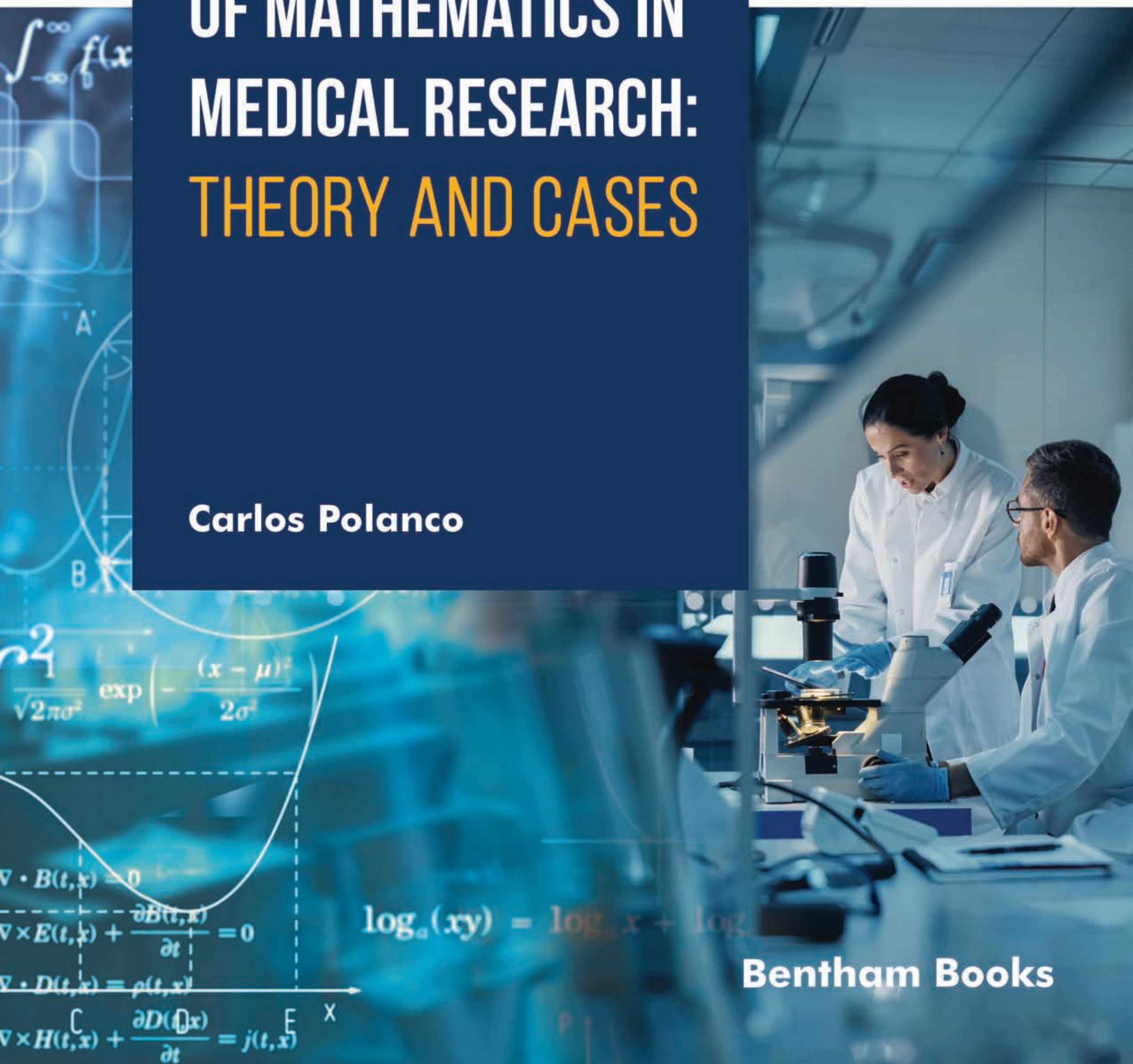
FUNDAMENTALS OF MATHEMATICS IN MEDICAL RESEARCH: THEORY AND CASES

Carlos Polanco



Bentham Books

Fundamentals of Mathematics in Medical Research: Theory and Cases

Authored by

Carlos Polanco

Department of Electromechanical Instrumentation
Instituto Nacional de Cardiologia Ignacio Chavez
Mexico City, Mexico

&

Faculty of Sciences
Universidad Nacional Autonoma de Mexico
Mexico City, Mexico

Fundamentals of Mathematics in Medical Research: Theory and Cases

Author: Carlos Polanco

ISBN (Online): 978-981-5223-13-2

ISBN (Print): 978-981-5223-14-9

ISBN (Paperback): 978-981-5223-15-6

First published in 2024.

Bentham Science Publishers Pte. Ltd.
80 Robinson Road #02-00
Singapore 068898
Singapore
Email: subscriptions@benthamscience.net

CONTENTS

FOREWORD I

The comprehensive guidebook “Fundamentals of Mathematics in Medical Research” bridges the distance between mathematics and the field of medical research. This enthralling investigation offers a fresh perspective on the fundamental mathematical concepts that undergird rigorous medical analysis and decision-making.

This book provides medical researchers with a firm foundation for modeling and comprehending complex systems by delving into the complex world of functions and analyzing their behavior in both single and multiple variables. It seamlessly transitions to probability, enabling readers to comprehend the inherent uncertainty in medical data and providing them with the tools to make informed decisions.

The book chapter, Estimation and Decision Theory, reveals the complexities of statistical inference and optimal decision-making. With an emphasis on Nonparametric Statistical Methods, the text emphasizes approaches to data analysis that are robust and adaptable to a variety of medical research circumstances.

In addition, “Fundamentals of Mathematics in Medical Research” ventures on an illuminating journey through Correlation Theory, shedding light on the intricate relationships between variables and their influence on medical research outcomes. It deftly examines the art of curve fitting, allowing researchers to uncover concealed patterns and trends in data, thereby enhancing their ability to make precise predictions.

With a distinct emphasis on Multivariate Analysis of Variance, this book equips medical researchers with effective techniques for dissecting complex data sets, thereby enabling the extraction of insightful information. The paper concludes by exploring Discrete-Time Markov Chains, illuminating the intriguing world of stochastic processes and their applications in medical research contexts.

“Fundamentals of Mathematics in Medical Research” provides a comprehensive framework for medical researchers to navigate the complexities of data analysis, modeling, and decision-making through its insightful and accessible approach.

This invaluable resource not only equips readers with mathematical tools, but also instills confidence in their capacity to analyze medical research data critically, nurturing a deeper comprehension of its implications for patient care and scientific progress.

Vladimir N. Uversky
University of South Florida
Tampa, FL, USA

FOREWORD II

"Fundamentals of Mathematics in Medical Research" is an indispensable resource that bridges the divide between mathematics and its applications in medical research. This exhaustive book examines a vast array of mathematical concepts and techniques that are essential for conducting rigorous and evidence-based medical research.

In today's data-driven healthcare environment, medical researchers require a firm foundation in mathematics to analyze and interpret complex data sets, make informed decisions, and draw meaningful conclusions. This book is a valuable resource for both aspiring and established medical researchers, equipping them with the mathematical tools necessary to navigate the complexities of their work.

The book begins by establishing the groundwork by examining the functions of a single variable and multiple variables. Through lucid explanations and illustrative examples, the reader is introduced to essential mathematical functions and their applications in medical research. Understanding these concepts is essential for accurately modeling medical phenomena and conducting statistics.

Probability, a linchpin of contemporary statistics, is an additional focus of this volume. The authors highlight the fundamentals of probability theory and instruct readers on how to quantify uncertainty and make sound decisions based on statistical inference. By mastering probability theory, medical researchers acquire a potent instrument for assessing the probability of events and making confident predictions.

Estimation and Decision Theory play a crucial role in medical research, and this book provides a thorough study of on these subjects. The reader will learn numerous estimation techniques, such as point estimation and interval estimation, as well as optimization-based decision-making strategies. With these instruments, researchers can make decisions based on data and derive valid conclusions from their studies.

When coping with data that does not conform to traditional parametric assumptions, non-parametric statistical methods are indispensable. This book covers the fundamentals of non-parametric statistics and equips readers with alternative methods for analyzing and interpreting data sets with complex or irregular distributions.

Correlation Theory investigates the relationships between variables, enabling researchers to recognize associations and dependencies in their data. This book provides a comprehensive overview of correlation analysis techniques, allowing medical researchers to identify meaningful relationships between variables and disclose crucial insights.

The potent mathematical technique of curve fitting is used to model and analyze empirical data. The authors provide a comprehensive introduction to curve fitting methods, enabling readers to apply mathematical models to actual medical data. This enables the characterization and prediction of medical phenomena, aiding researchers in their pursuit of effective diagnostic, therapeutic, and preventative strategies.

Multivariate Analysis of Variance (MANOVA) is an indispensable statistical technique for examining the relationship between multiple variables. This book enables medical researchers to analyze complex data sets with multiple factors and evaluate the significance of their findings by investigating the principles and applications of MANOVA.

The book concludes with an in-depth examination of discrete-time Markov chains. These mathematical models capture the dynamics of sequential events and transitions, making them indispensable for comprehending and predicting medical processes and outcomes. By analyzing discrete-time Markov chains, medical researchers can model the behavior of medical phenomena over time and make accurate forecasts.

Thomas Buhse
Universidad Autonoma del Estado de Morelos
Cuernavaca Morelos, Mexico

PREFACE

“Fundamentals of Mathematics in Medical Sciences” is a book geared toward researchers and students in the medical field. The most significant mathematical issues needed for academic and research domains are attempted to be addressed simply. To make it easier for readers to understand these key ideas, complete solutions are given for all examples, exercises, and case studies.

The book claims to be more than just a list of topics. It introduces each topic as a component of a continuous stream of mathematical ideas that are clarified with simple examples and shows how useful mathematics is in medicine.

The concept of real-valued functions of one or more variables, their parts, and their geometric representation in the plane and in space are all thoroughly examined in the first section of this book. Understanding these ideas is necessary for understanding the following subjects.

The second section discusses Independent Probability, Conditional Probability, Bayes’ theorem, and the fundamental techniques of Parametric and Nonparametric Statistics.

Using correlation theory, the Least Squares approach, and their analytical validation for the general case of polynomial real-valued functions, the third section of the book focuses on recovering real-valued functions of one or more variables. This is similar to how the first section of the book was structured.

The Discrete-Time Markov Chain Model and the Hidden Markov Model are both thoroughly examined in the final section, which is devoted to advanced multivariable analytic subjects. Following comprehension, the reader will find them to be of great use in research.

The appendix of this paper provides Fortran-90 programs, Python scripts, and Linux scripts for reproducing the offered information. According to the TIOBE programming community index, Fortran is utilized in scientific computing in addition to C and C++ on supercomputer platforms and High-Performance Computing clusters as of July 2022. The information offered here should prove helpful and motivating to readers who are interested in the principles of this subject.

The National Institute of Cardiology “Ignacio Ch´avez” and the Faculty of Sciences at the National Autonomous University of Mexico have to be thanked for giving instances and examples, respectively.

Carlos Polanco
Department of Electromechanical Instrumentation
Instituto Nacional de Cardiologia Ignacio Chavez
Mexico City, Mexico

&

Faculty of Sciences
Universidad Nacional Autonoma de Mexico
Mexico City, Mexico

ACKNOWLEDGEMENTS

I would like to thank all those whose recommendations made the publication of this ebook possible.

Part I

SINGLE AND MULTIVARIABLE FUNCTIONS

The first part of this text introduces the reader to the real-valued functions of a single variable and those of multiple variables. It investigates their transformations as well as the various categories of functions in the plane and space. It describes in detail the most frequently employed operations, the benefits of utilizing vector-valued functions, and their geometric representation in the plane and space. The section is self-contained, allowing a reader unfamiliar with the topic to continue reading.

CHAPTER 1

Functions using a Single Variable

Abstract: In this chapter, we will delve into a variety of fundamental subjects related to the study of real functions and the transformations of such functions. To begin, we will investigate the absolute value, as well as the absolute value function, and examine the behavior and attributes of each of these concepts. Next, we will delve deeper into the world of functions, concentrating on the functions that are associated with a single variable and the graphical representations of these functions. As we move further, we will investigate the most important real-valued functions, such as polynomial functions and piecewise-defined functions, amongst others. Because of these functions, we are able to acquire a deeper comprehension of transformations and how they have an effect on function graphs. In this lesson, we will investigate the scaling and translating transformations, focusing on how they affect the features of functions. Examples and activities are presented throughout the chapter to help readers better understand the fundamental ideas behind graphical representation and function analysis.

Keywords: Absolute value, absolute value function, functions, functions of one variable, graph f, main real-valued functions, piecewise real functions, polynomial real functions, real-valued functions, scaling transformations, transformations, translation transformations.

1.1. Introduction

This chapter introduces an essential mathematical operator for the topics we will address *i.e.* the real-valued function f that makes it possible to relate two variables, x, and y. Assume that the body temperature T depends on the ambient pressure P, it can be denoted as $T(P)$; if the temperature is experimentally verified to be equivalent to the square of the pressure minus four units, then it will be denoted as $T(p) = p^2 - 4$.

Let's take a closer look at the usefulness of this operator focusing on the relationship between body temperature and ambient pressure, to see how the former is affected by the latter. Taking some data we have, (p,t) $= (12\text{hPa}, 140°\text{C})$, (p, t) $= (23\text{hPa}, 525°\text{C}), \cdots,$ (p, t) $= (34\text{hPa}, 1152°\text{C})$. The function that represents these values is $T(p) = p^2 - 4$.

It is important to note that with this function $T(p)$, we can calculate any value, furthermore, other data will not be required, to find higher, lower, or intermediate values at any interval. Of course, this is an example and it is not accurate. For a real study, it will be necessary to gather more information to be precise, as it only shows how these functions can be used.

Carlos Polanco

1.2. Functions

A function f is a rule that relates two sets. Its correct definition depends, on relating each element a of the set A, *i.e.* $a \in A$, with only one element b of the set B. The association of set A with set B in function f is denoted as $f : A \to B$.

Definition 1.1. A **function** f, in a general sense, is a **well defined rule** that relates two sets $f : A \to B$, where each element in set A corresponds to a **unique** element in set B.

Example 1.1. (i) Give a non-numerical example of a function f. (ii) Give a numerical example of a function f. (iii) Give an example of a relationship that is **not** a function. (iv) Discuss (i –iii).

Solution 1.1. (i) Let some birds be in set $A = \{\text{ostrich}, \text{myna}, \text{parrot}\}$ and some letters in set $B = \{\text{o}, \text{m}, \text{p}\}$. Function f relates each bird in set A with a letter in set B with this rule: the initial letter of the name of the bird will be associated with the correspondent letter in set B. (ii) Let set $A = \{1,2,3,4\}$ and set $B = \{1,4,9,16\}$, where the function $f : A \to B, x^2$. (iii) Let set $A = \{1,2,3,4\}$ and set $B = \{\text{a}, \text{b}, \text{c}, \text{d}, \}$, where function f has this rule: give each element of set A a letter of set B at random. (iv) The designation rule in (i) is well defined and avoids mistakes when relating both sets. In the designation rule of function (ii), the funcion $f(x) = x^2$ connects each element of set A with its square in set B. Finally, the relation in (iii) is not a function since it is not possible to link each element in set A with an element of set B, as the rule is vague.

A function that associates an element of the real number set $\mathbb{R}$ with another number of the same set i.e., $\mathbb{R} \to \mathbb{R}$ is a **real function of one variable** or a **real-valued function of one variable**.

Example 1.2. (i) Give an example of a real-valued function of one variable. (ii) Draw its graph.

Solution 1.2. (i) $f(x) = 3x - 5$ represents a line since the exponent of the variable x is 1. Two points in this line are $(0, -5)$ and $(3, 4)$. An equivalent representation of this function is $f : \mathbb{R}s \to \mathbb{R}, 3x - 5$. To obtain these two points, we evaluate the function at the values 0 and 3. When the function represents a line, it is enough to determine two points to draw it on the x/y plane. (ii) See Fig. (**1.1**).

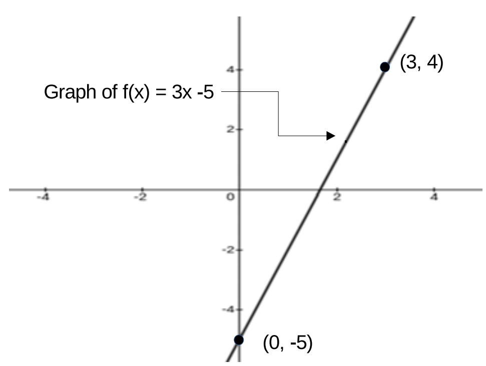


Fig. (1.1): Geometrical representation of a real-valued function of one variable $f(x) = 3x - 5$. Graph generated by [1].

According to the example Ex. (**1.2**), the slope m of the line is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, where $(x_1, y_1) = (0, -5)$, y $(x_1, y_1) = (3, 4)$, so $m = 9/3 = 3$. To have this slope in degrees we calculate $\tan^{-1} m = \tan^{-1} 3 \approx 71$ grades.

1.3. Properties of Real-Valued Functions

The real-valued functions of one variable can be handled with the operations of addition, subtraction, multiplication, division and exponentiation that create new real-valued functions of one variable, *i.e.* if $f(x) = x^2$ and $g(x) = 2x - 1$, then $h(x) = f(x) + g(x) = x^2 + 2x - 1$ or $m(x) = f(x)g(x) = x^2(2x - 1) = 2x^3 - x^2$.
If $f(x)$ and $g(x)$ are real-valued functions of one variable, the following rules (Def. (**2.3**)) are met.

Rule (i) $h(x) = f(x) + g(x)$

Rule (ii) $h(x) = f(x) - g(x)$

Rule (iii) $h(x) = f(x)g(x)$

Rule (iv) $h(x) = \dfrac{f(x)}{g(x)}$ where $g(x) \neq 0$, for all $x \in \mathbb{R}$

Rule (v) $h^{\alpha}(x) = [f(x) \bigodot g(x)]^{\alpha}$, $\alpha \in \mathbb{R}$

Example 1.3. Give an example of each operation (Def. (**2.3**)) if $f(x) = x + 1$ y $g(x) = x - 2$.

Solution 1.3. (i) $h(x) = f(x) + g(x) = 2x - 1$. (ii) $h(x) = f(x) - g(x) = 3$. (iii) $h(x) = f(x)g(x) = (x+1)(x-2) = x^2 - x - 2$. (iv) $h(x) = \dfrac{f(x)}{g(x)} = \dfrac{x+1}{x-2}$, for $x \neq 2$. (v) $h^2(x) = [f(x) + g(x)]^2 = (2x-1)^2 = 4x^2 - 4x + 1$.

1.4. Transformations of Real-Valued Functions

Some actions can move functions, change their size, rotate them, or reflect them on another axis. These actions are called transformations and they are useful in adapting, a function to the values to be represented.

1.4.1. Translation

It moves an entire real-valued function of one variable in a specific direction. To perform this movement on a function $f(x)$, we will have to add or substract a real number to the original function, i.e., $f(x) \pm \alpha$, where $\alpha \in \mathbb{R}$ to move it to the right or left and $f(x \pm \alpha)$ to move it upwards or downwards.

Example 1.4. (i) Translate function $f(x) = x^3$ three units to the right. (ii) Translate function $f(x) = x^3$ three units to the left. (iii) Translate function $f(x) = x^3$ two units upwards. (iv) Translate funtion $f(x) = x^3$ two units downwards. (v) Graph (i-iv).

CHAPTER 2

Functions of Several Variables

Abstract: This chapter reviews the real-valued functions on space, with fully solved examples that show the application and usefulness of their components to deal with practical problems. Then, we see the types of functions that the reader will often find in academic and research activities. Later, another family of functions called Vector-Valued Functions is defined and its usefulness in solving application problems is shown with examples. Finally, both types of functions are generalized for higher-dimension spaces.

Keywords: Absolute value, functions, high-dimensional RVF, main RVF, polynomial RVF, several variables functions, trigonometry RVF, ute-value real-valued functions, vector-valued functions.

2.1. Introduction

Here, we introduce an extension of the function. A function of several variables f relates three or more variables. For instance, body temperature T depends on the ambient pressure P and the day of the week D, denoted as $T(P,D)$. If the temperature is equivalent to the square of the pressure minus the day of the week, it will be denoted as $T(p,d) = p - d$.
With the relationship between body temperature, day of the week, ambient pressure, and the data (p,t,d) $=$ (12hPa, 140°C, Monday), (p, t, d) $=$ (23hPa, 525°C, Tu esday), $\cdots$, (p, t, d) $=$ (34hPa, 1152°C, Friday), we get a function representing these values $T(p,d) = p - d$.
With this function $T(p,d)$, we can calculate lower, higher, or intermediate values at any region bounded by the parameters p and d without needing more information. This approach is far from being accurate, as we will be needing more data to improve accuracy, however, it shows the usefulness of these functions.
Later, we will review and give some examples of this type of function.

2.2. Functions

The function of two variables f associates two or more sets, so each element a in set A and each element b in set B are associated only with one element c in set C. The association of sets A and B with set C is denoted as $f : (a,b) \in \mathbb{R}^2 \to C$.

From this statement, we can characterize the function f of several variables.

Definition 2.1. A **function of several variables** [2], is a **well-defined rule** f that relates two sets $f : (a,\cdots,b) \in \mathbb{R}^n \to C$, where each element in set A and each element in set B correspond to a **unique** element in set C.

Example 2.1. (i) Provide an example of a two-variable function f. (ii) Provide an example of a three-variable function f. (iii) Discuss (i –ii).

Carlos Polanco

Solution 2.1. (i) Let $A = \{\text{ostrich}, \text{myna}, \text{parrot}\}$, set $B = \{\text{o}, \text{m}, \text{p}\}$, the function f relates each element $(a,b) \in \mathbb{R}^2$ and a number in set $C = \{1,2,\cdots,n\}$, with the rule of f that associate $(bird, letter)$ with a consecutive number in set C. (ii) Let set $A = \{1,2,3,4\}$, set $B = \{1,4,9,16\}$, and set $C = \{0,-2,-6,-12\}$, where the real-value function of three variables is $f : (a,b,c) \in \mathbb{R}^2 \to D, a-b-c$. (iii) The assignation rule of (i) is well-defined and avoids confusion when connecting both sets. In the assignation rule of (ii), function $f(x,y,z) = x-y-z$, connecting each element (a,b,c) with the set D.

An RVF that relates an element of the set $\mathbb{R}^2$ with an element of the set $\mathbb{R}^2 \to \mathbb{R}$ is a **real function of two variables**..

Example 2.2. (i) Provide an example of an RVF. (ii) Plot the graph.

Solution 2.2. (i) $f(x) = \sin x \cos y$ is a wave with peaks and valleys expressed in 3D space $\mathbb{R}^3$. Two points in this graph are $(\frac{\pi}{2}, \pi, -1)$ and $(\pi, 0, 0)$. An equivalent representation for the RVF of the two variables mentioned is $f : \mathbb{R}^2 \to \mathbb{R}, \sin x \cos x$. We obtain these two points by evaluating the RVF of two variables with the values $(\frac{\pi}{2}, \pi)$ and $(\pi, 0)$. (ii) See Fig. (**2.1**).

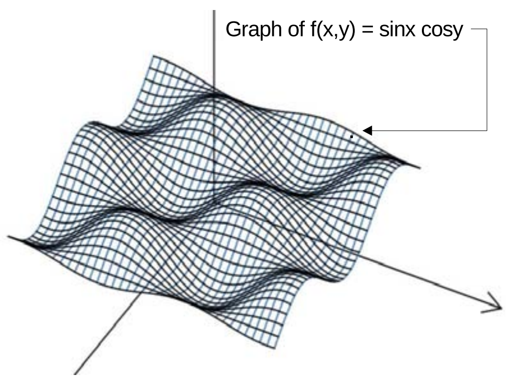


Fig. (2.1): Geometrical representation of a two variable RVF $f(x,y) = \sin x \cos y$. Plotted by [1].

Plotting an RVF of several variables is not a simple task, it requires the observation of several graphs to identify the family these functions belong to and have a previous idea of the region that is convenient to use.

2.3. Properties of the RVF

The RVF of several variables can be the subject of the addition, subtraction, multiplication, division, and exponentiation operations, creating new RVF of several variables, *i.e.* if $f(x) = x^2$ and $g(x) = 2x-1$, thus $h(x) = f(x) + g(x) = x^2 + 2x - 1$, or $m(x) = f(x)g(x) = x^2(2x-1) = 2x^3 - x^2$.
If $f(x_1 \cdots, x_n)$ and $g(x_1, \cdots, x_n)$, then they are the RVF of several variables that satisfy these rules (Def. (**2.3**)).

Rule **1**. $h(x_1,\cdots,x_n) = f(x_1,\cdots,x_n) + g(x_1\cdots,x_n)$

Rule **2**. $h(x_1,\cdots,x_n) = f(x_1,\cdots,x_n) - g(xi_1,\cdots,x_n)$

Rule **3**. $h(x_1,\cdots,x_n) = f(x_1,\cdots,x_n)g(x_1,\cdots,x_n)$

Rule **4**. $h(x_1,\cdots,x_n) = \dfrac{f(x_1,\cdots,x_n)}{g(x_1\cdots,x_n)}$ where $g(x_1,\cdots,x_n) = 0$, for all $x \in \mathbb{R}$

Rule **5**. $h^{\alpha}(x_1,\cdots,x_n) = [f(x_1,\cdots,x_n) \bigodot g(x_1,\cdots,x_n)]^{\alpha}$, $\alpha \in \mathbb{R}$

Example 2.3. Provide an example of each operation in Def. (**2.3**) if $f(x) = x+1$ and $g(x) = x-2$.

Solution 2.3. (i) $h(x) = f(x)+g(x) = 2x-1$. (ii) $h(x) = f(x)-g(x) = 3$. (iii) $h(x) = f(x)g(x) = (x+1)(x-2) = x^2-x-2$. (iv) $h(x) = \dfrac{f(x)}{g(x)} = \dfrac{x+1}{x-2}$, for $x = 2$. (v) $h^2(x) = [f(x)+g(x)]^2 = (2x-1)^2 = 4x^2-4x+1$.

2.4. RVF

Different RVF of several variables exist; they can be polynomial, exponential, trigonometrical, logarithmic, piecewise, and absolute value functions, among others.

2.4.1. Polynomial RVF

A polynomial RVF of several variables is a function whose terms are variables with exponents that are positive integers in the addition, subtraction, and multiplication operations.

Definition 2.2. In a polynomial RVF of several variables like Eq. (**2.1**), the exponents are $n \in \mathbb{N}$ and the coefficients are $a_i,\ b_j \in \mathbb{R}$.

$$f(x,y) = a_n x^n b_n y^n + a_{n-1}x^{n-1}b_{n-1}y^{n-1} + \cdots + a_2x^2b_2y^2 + a_1xb_1y + a_0b_0 \quad (2.1)$$

Remark 2.1. Set $\mathbb{N}$ has positive integer numbers, *i.e.* $\{1,2,3,\cdots,n\}$, and the **degree** of a polynomial is determined by the highest exponent.

Example 2.4. (i) Provide a 3rd degree polynomial RVF of several variables. (ii) Provide a 1st-degree polynomial RVF of several variables. (iii) Plot (i) and (ii).

Solution 2.4. (i) $f(x) = x^3+y$. (ii) $f(x) = x$.
(iii) See Figs. (**2.2**), (**2.3**).

Part II

INFERENCE BASED ON DATA

In the second part of this book, three fundamental ideas are presented: probability, conditional probability, and Bayes' theorem. This article focuses on the most frequent parametric and non-parametric approaches that are used to mathematically verify the experimental hypothesis. This section is self-contained, which enables the reader who is not familiar with the subject to continue their research.

CHAPTER 3

Probability

Abstract: In this chapter, we delve into the key concepts of probability theory and the most common statistical distributions. We begin by introducing Bayes' Theorem and its application in statistical inference. Next, we address the binomial distribution function and its usefulness in analyzing binary events and calculating probabilities. Then, we examine conditional probability and its importance in decision-making under uncertainty. Subsequently, we immerse ourselves in the main probabilistic functions, including the normal distribution function and its role in modeling natural phenomena. Furthermore, we study the Poisson distribution function, which is applied to situations where the probability of rare events occurring needs to be calculated. Finally, we analyze the general concept of probability and its interpretation in the context of statistical theory. Additionally, we present the Total Probability's Theorem, which allows for the calculation of the probability of an event based on conditional event information. In summary, this chapter provides a solid foundation for the fundamentals of probability and statistics, exploring key topics such as Bayes' Theorem, the binomial distribution function, conditional probability, main probabilistic functions, the normal distribution function, the Poisson distribution function, probability, and the Total Probability's Theorem.

Keywords: Bayes' theorem, binomial distribution function, conditional probability, main probabilistic functions, normal distribution function, poisson distribution function, probability, total probability's theorem.

3.1. Introduction

In the realm of uncertainty and chance, probability is an indispensable instrument for comprehending and modeling the surrounding phenomena. In this chapter, we will examine a variety of fundamental concepts and functions in the field of probability, establishing the foundation for rigorous and accurate analysis.

We will begin our voyage with Bayes' theorem, a mathematical gem that enables us to revise our beliefs in light of new information. Thomas Bayes formulated this theorem in the 18th century, giving us a powerful tool for calculating conditional probabilities with applications spanning from medicine to artificial intelligence.

The binomial distribution is a function that depicts the number of successes in a set of independent trials with a fixed success probability. This distribution is shown to be particularly useful when determining the probability of obtaining a specific number of successes in repeated experiments.

Conditional probability will also be a central theme in this chapter. We will investigate how to calculate the probability of an event A given that event B has already occurred, as well as how this concept of an event dependence aids our comprehension of complex phenomena.

We will investigate the principal probabilistic functions, including the normal distribution and the Poisson distribution, in the following sections. Due to its ability to model a vast array of natural and social phenomena, the former, also known as the Gaussian bell curve, is extensively utilized. The Poisson distribution, on the other hand, allows us to describe the probability of a number of events occurring in a

Carlos Polanco

given time interval, making it an indispensable instrument for analyzing systems with occurrence rates.
Finally, we will examine the fundamental concept of probability and the theorem of total probability. This theorem enables us to calculate the probability of an event based on its various possible outcomes, providing a methodical and consistent approach to solving complex problems.
This chapter will delve into each of these topics, providing illustrative examples and plain explanations to help you develop a firm grasp of probabilistic foundations. These tools will enable you to confidently confront the challenges posed by uncertainty in various knowledge and decision-making domains. Let's embark on an adventure into the intriguing realm of probability!

3.2. Probability

Definition 3.1. Probability [3, 4, 5] P is the measure of the possibility that a **random event** A_i has occured. This **measure** is expressed in a range that can be at least 0 if the possibility is impossible to occur, or at the most 1 if there is total certainty that the event has occured.

Example 3.1. (i) Give an example of an impossible event. (ii) Give an example of a certain event.

Solution 3.1. (i) Number eight shown in a throw of the dice when it only has the numbers 1 to 6. (ii) Any number of the dice shown in a throw.

The probability $P(A)$ is usually expressed as a quotient, whose dividend is the **number** of random events A_i of the phenomenon measured, and the divisor is the total **number** of the possible random events. The set Ω that has all possible random events is called **sample space**.

Example 3.2. Let a throw of a dice. (i) Determine the probability of getting an even number on the first throw. (ii) Determine the probability to get an odd number. (iii) Determine the probability to get number three.

Solution 3.2. (i) $P(A) = \frac{3}{6} = \frac{1}{2}$. (ii) $P(A) = \frac{3}{6} = \frac{1}{2}$. (iii) $P(A) = \frac{1}{6}$.

3.3. Conditional Probability

Conditional probability is a specific measure of an event in which the occurrence is measured on a different set than the universal set U (Def. (**3.2**)), and then it is substituted by another for which we know the certainty that it already exists.

Definition 3.2. Let a sample space Ω, a random event A for $P(A) > 0$, and an arbitrary random event $B \in \Omega$. The probability that a random event B occurs given that the random event A also occurs (in symbols $P(B|A)$), is defined as Eq. (**3.1**).

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{3.1}$$

It is important to understand the significance of the operator "|" as a restriction of the set where the operation $A \cap B$ acts. Note in Fig. (**3.1**) that the operation $A \cap B$ acts on the set U, while in Fig. (**3.2**), the same operation acts **only** on the set B. This happens when you limit the probability of set A over set B *i.e.* $P(A|B)$, set U is reduced to set B.

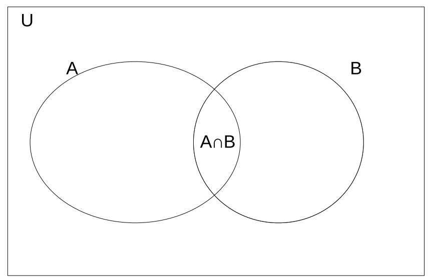


Fig. (3.1): Geometrical representation of $A \cap B$ over set U.

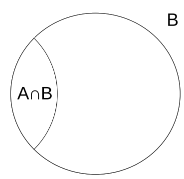


Fig. (3.2): Geometrical representation of $A \cap B$ over set B.

Example 3.3. With the information of Table (**3.1**), determine (i) $P(B)$. (ii) $P(X \cap B)$. (iii) $P(X|B)$. (iv) $P(B|X)$. (v) Discuss the results in (iii and iv). (vi) Discuss (ii) compared to (iii).

Table 3.1: Values of species A and B versus environmental factors

	Factor X	Factor Y	**Total**
Species A	10,000	20,000	30,000
Species B	30,000	15,000	45,000
Total	40,000	35,000	75,000

Solution 3.3. (i) $P(B) = \dfrac{45,000}{75,000} = 0.60$ (ii) $P(X \cap B) = \dfrac{40,000}{75,000} = 0.53$

CHAPTER 4

Estimation and Decision Theory

Abstract: This chapter covers the basic concepts of inferential statistics and hypothesis testing. The section commences with an explanation of the confidence intervals for the population mean and population standard deviation. Also discussed are the concepts of cut-off, Type I and Type II errors, false positives, and false negatives. We investigate performance metrics including sensitivity, specificity, true positive rate, and true negative rate. In addition, receiver operating characteristic (ROC) curves are introduced and their usefulness in assessing test performance is emphasized. In addition, binomial and normal distribution tests are discussed. This chapter provides a solid foundation for comprehending the principles and practical applications of inferential statistics in decision-making and the interpretation of scientific results.

Keywords: Confidence interval for a population mean, confidence interval for a population standard deviation, cut-off, false negatives, false positives, sensitivity, specificity, statistical hypothesis and significance, receiver operating characteristic curve, ROC curve, tests involving binomial distributions, tests involving normal distributions, true negatives, true negative rate, true positives, true positive rate, type I error, type II error.

4.1. Introduction

Data-driven decision-making is crucial in scientific research and various fields of study. However, data alone do not always provide definitive answers. It is in this context that concepts such as confidence intervals, statistical tests, and Type I and Type II errors come into play.

In this captivating chapter, we will delve into the fascinating world of statistical inference, where we will explore a range of topics essential for understanding and quantifying the certainty and uncertainty associated with our analyses. We will examine, everything from constructing confidence intervals for population mean and standard deviation to interpreting receiver operating characteristic (ROC) curve and conducting tests involving binomial and normal distributions.

We will embark on our journey by introducing the concept of confidence interval, which allows us to estimate the plausible range within which the true value of a population lies. We will explore how to calculate confidence intervals for population mean and standard deviation and analyze how the choice of confidence level and sample size affect the precision of our estimates.

As we delve deeper into statistical tests, we will immerse ourselves in the world of Type I and Type II errors. We will discuss the meaning of these errors and how they relate to statistical significance and null and alternative hypotheses.

To evaluate the effectiveness of diagnostic and classification tests, we will examine sensitivity, specificity, false positive rate, and false negative rate. Furthermore, we will introduce the powerful tool of ROC curve, which allows us to visualize and compare the performance of different tests based on sensitivity and specificity.

Carlos Polanco

Finally, we will delve into tests involving binomial and normal distributions, where we will learn how to make inferences about proportions and differences in means.

Throughout this chapter, we will discover how these fundamental statistical concepts intertwine to provide us with a deeper understanding of certainty and uncertainty in our analyses. Get ready for an exciting journey to the essence of statistical inference and its real-world application.

4.2. Statistical Hypothesis and Significance

In many studies, a parameter is used to decide whether or not to accept the hypothesis. This method is called **statistical hypothesis**. This is one of the most useful parts of statistical inference, since many studies that involve making decisions can be set up as hypothesis-test experiments [6, 7].

In short, a **statistical hypothesis** is a guess based on a parameter of a group.

Suppose we need to know the speed with which a tank of water is emptied, so we will have to focus on the average exit speed, particularly if the average speed of exit is 80 cm/s or not. This is called the **null hypothesis** and can be written as $H_0 : \mu = 80$. On the other hand, the statement $H_1 : \mu \neq 80$ is called an **alternate hypothesis**.

A **null hypothesis** H_0 assumes that a determined parameter, such as μ, is not bias.

An **alternative hypothesis** (abbreviated as H_1) is formulated whenever the **null hypothesis** (abbreviated as H_0) is not validated. This is the hypothesis in which the researcher seeks to determine whether or not $H_1 : \mu > 80$, $H_1 : \mu 80$, or $H_1 : \mu \neq 80$ is true.

4.3. Errors Type I and Type II

When conducting a statistical hypothesis test, it is possible to encounter two types of errors: **Type I error** and **Type II error**. The two errors exhibit an inverse relationship and are influenced by the significance levels α and β. The determination of the margin of error is a crucial aspect during the design phase of the test.

(A) **Type I error**. If the null hypothesis H_0 is rejected when it is true, the error is of type I. The probability to make this type of error will depend on the significance level α, determined when designing the experiment for the hypothesis test.

A significance level of $\alpha = 0.01$, for instance, indicates that there is a 1% chance of error when rejecting the null hypothesis. To reduce this risk, a lesser α value must be selected. However, selecting a lesser value for α reduces the likelihood of detecting a difference, if one exists.

(B) **Type II error**.Type II error occurs when the null hypothesis H_0 is not rejected when it is false. The probability of committing a type II error is proportional to the significance level (β). To reduce the likelihood of committing this error, we must increase the sample size.

If the alternative hypothesis is true, then the probability of rejecting the null hypothesis is equal to $1 - \beta$.

The results of a statistical test are probable values, therefore, the experimentalist will have to choose to minimize one type of error neglecting the other.

Example 4.1. An experimentalist wants to compare the effectiveness of two vaccines for COVID-19. The null and alternative hypotheses are:

(A) Null Hypothesis $H_0 : \mu_1 = \mu_2$, *i.e.* both vaccines have the same effectiveness.

(B) Alternative hypothesis $H_1 : \mu_1 \neq \mu_2$, *i.e.* both vaccines **do not** have the same effectiveness.

(i) Provide an explanation for both hypotheses. (ii) Please include a table that details both of the mistakes. (iii) Using the intersection of two normal distribution functions, draw both the errors that were made. (iv) Is it possible to reduce mistakes I and II while doing so simultaneously?

Solution 4.1. (i) A type I error occurs when the experimentalist disregards a null hypothesis H_0 and concludes that the two COVID-19 vaccines are different, when in fact, they are not. If the vaccines have the same effectiveness, the experimentalist might consider that this error is not so important, as in any case, the subjects will benefit with the same level of effectiveness regardless of what vaccine they get. In the event that a type II error occurs, the experimentalist fails to ignore the null hypothesis H_0 when it would have been appropriate to do so. This means that the experimentalist concludes that the COVID-19 vaccines are the same, when in fact, they are different. This error can put lives at risk if the less effective vaccine is sold to the public instead of the most effective one.

(ii) See Table (**4.1**).

Table 4.1: Type I and type II errors.

Null hypothesis H_0 is...	True	H_0 False
Rejected H_0	False positive. Type I error. Probability = α	Correct decision. True positive. Probability $1 - \beta$
Not rejected H_0	Correct decision. True negative. Probability $1 - \alpha$	False negative. Type II error. Probability = β

(iii) See Fig. (**4.1**).

CHAPTER 5

Non-Parametric Statistic Methods

Abstract: In this chapter, we will explore various methods and statistical tests used in hypothesis validation and sample comparison. We will begin with an introduction to the formulation of scientific and statistical hypotheses and discuss the determination of the appropriate sample size. Next, we will delve into the analysis of independent and related samples, examining techniques, such as the one-sample binomial test, one-sample method, one-sample runs test, two-sample chi-squared test, two-sample Kolmogorov test, two-sample runs test, and two-sample U test. These tools will provide researchers and scientists with the ability to rigorously evaluate their hypotheses and compare samples in a reliable manner.

Keywords: Hypothesis, independent samples, methods, one sample binomial test, one sample method, one sample runs test, related samples, samples, sample size determination, scientific hypothesis, statistical hypothesis, two sample method, two sample Chi-squared test, two sample Kolmogorov test, two sample runs test, two sample U test.

5.1. Introduction

In the world of scientific research, the formulation and subsequent testing of hypotheses have become a fundamental pillar for reaching valid and trustworthy conclusions. In this fifth chapter, we will investigate various methods for testing hypotheses and concepts pertaining to independent samples. These methods enables us to evaluate claims and assess various data sets to determine whether there are any significant differences between them.

We will begin with a discussion of the theoretical foundations, laying the groundwork for understanding the various methodologies for testing hypotheses. We will examine scientific and statistical hypotheses, in addition to the significance of sample size determination in study design. Understanding how to design a study with a sufficient sample size is essential in ensuring that the results are representative and generalizable.

Following this, we will examine specific techniques for analyzing a single sample. We will examine the one-sample binomial test and the one-sample runs test, which enables us to evaluate the success rate and randomness of a dataset. When attempting to confirm or refute a claim about a binary variable or when searching for evidence of non-random patterns in the data, these techniques are particularly useful.

We will then delve into the analysis of relevant samples. These samples are paired or connected in some manner, enabling more accurate comparisons. Here, we will discuss a variety of statistical tests, such as the two-sample U test and the two-sample chi-squared test, which enables us to determine whether there are statistically significant differences between two related or paired groups.

Finally, we will investigate methods for testing hypotheses for independent samples in which there is no explicit relationship between the observations of the two groups being compared. The two-sample Kolmogorov test, and the two-sample runs test will be analyzed to determine whether there are statistically significant differences in the distribution or data patterns of two independent groups.

Carlos Polanco

5.2. Samples

There are two types of samples to be considered depending on the information you have of the population studied [10, 11].

5.2.1. Related Samples

A related sample is extracted from a population known for a specific feature or characteristic, for instance, the undergraduate students of linear algebra in a university. If it is required to verify a hypothesis with this population, a related sample will be used since all students have the same training

5.2.2. Independent Samples

An independent sample is taken from a population that has no relation with the hypothesis studied, for instance, a statistical study where the participants are taken at random so nothing is known a priori about them.

5.2.3. Sample Size Determination

The sample size determination needs to be inferred from an algorithm, though in many cases, this is subject to the budget for the statistical study. However, it is always advisable to apply one of the different formulas that already exist.
For simplicity, we suggest the following sample size n (Eq. (**5.1**)).

$$n = \frac{Z_\alpha^2 N p q}{e^2(N-1) + Z_\alpha^2 p q} \tag{5.1}$$

where

(i) N is the size of the sample.

(ii) Z_α is a constant that depends on the level of trust α we assign. If $\alpha = 80\%$, then $Z_\alpha = 1.28$, if $\alpha = 90\%$, then $Z_\alpha = 1.65$, if $\alpha = 95\%$, then $Z_\alpha = 1.96$, or if $\alpha = 99\%$, then $Z_\alpha = 2.57$.

(iii) e is the sample error determined by the experimentalist. If 1% is estimated, then $e = 0.01$, or if 9% is estimated, then $e = 0.09$.

(iv) p is the proportion of elements in the population, that have the characteristics studied. If the value is unknown, it is assumed that $p = q = 0.5$. So, q is the proportion of elements that do not have that characteristic $q = 1 - p$.

Remark 5.1. Taking a sample of the studied population instead of the total population, it will always be related to the real possibility of selection. If it is feasible to take the total population, it will always be more convenient. However, if this is not possible, it is advisable to estimate the size of the sample.

Example 5.1. Give the sample size for a population of 500,000 consumers of a beverage brand, where the experimentalist gives a trust level of 95%, an error margin of 3%, and the probability of the event p is unknown.

Solution 5.1. From Eq. (**5.2**), the sample size is 1064.

$$n = \frac{Z_{\alpha}^{2} Npq}{e^{2}(N-1) + Z_{\alpha}^{2} pq} = \frac{1.96^{2}(500000)0.50^{2}}{0.03^{2}(499999) + 1.96^{2}0.50^{2}} = 1064.8 \tag{5.2}$$

5.3. Hypothesis

There are two hypotheses an experimentalist must assume when addressing the viability of a hypothesis.
The first is to raise the scientific hypothesis and then validate the statistical hypothesis. If the statistical hypothesis is valid, the scientific hypothesis will be valid; otherwise, the scientific hypothesis will be rejected (Fig. (**5.1**)).

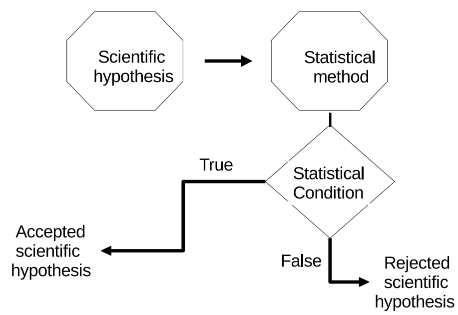


Fig. (5.1): Scientific and statistical hypothesis relation.

5.3.1. Scientific Hypothesis

This is the first hypothesis that comes from the observation of a natural event. It is the assumption resulting from that observation.

Example 5.2. Give an example of a scientific hypothesis.

Solution 5.2. An experimentalist observes a learning deficiency in university students, particularly in those whose mother was absent from the family. A possible scientific hypothesis would be, is the absence of the mother at an early age affecting the learning process?

Remark 5.2. The scientific hypothesis is explained in terms of the phenomenon studied.

5.3.2. Statistical Hypothesis

A statistical hypothesis is the conclusive condition of the statistical method chosen *i.e.* if $p < \alpha$ H_0 is accepted, otherwise H_t is accepted.

Example 5.3. From Problem (**5.2**), define a statistical hypothesis.

Part III

ANALYSIS BY REGRESSION

In the book's third part, readers are introduced to the concepts of correlation theory and curve fitting, two pillars of mathematical modeling. In this first chapter, we will go over the basics of graphic correlation and look at some examples. In the second chapter, you will learn how to use the Least Squares approach to build linear functions, nonlinear functions, and weighted functions. Finally, the Logistic Regression Model, a technique within the broader subject of regression analysis, is examined in detail.

CHAPTER 6

Correlation Theory

Abstract: In this chapter, we examine the nature of correlations in Euclidean spaces, focusing on the two-dimensional space $\mathbb{R}^2$ and the three-dimensional space $\mathbb{R}^3$. We begin by exploring linear correlations in $\mathbb{R}^2$, where we analyze calculation techniques and association measures to quantify the relationship between two continuous variables. Next, we delve into multiple correlations in $\mathbb{R}^3$, examining how several variables can be related simultaneously and how their strength and direction can be jointly measured. Subsequently, we address non-linear correlations in $\mathbb{R}^2$, expanding the focus beyond traditional linear relationships. We explore advanced methods and techniques for detecting and measuring non-linear correlations, allowing us to capture complex and non-linear patterns in the data. Furthermore, examples of practical applications are discussed where the presence of non-linear correlations is crucial for analysis and decision-making.

Keywords: $\mathbb{R}^2$ space, $\mathbb{R}^3$ space, linear correlation on $\mathbb{R}^2$, multiple correlation on $\mathbb{R}^3$, non-linear correlation on $\mathbb{R}^2$, Pearson's correlation factor, Spearman's correlation factor.

6.1. Introduction

When we are studying the behaviour of a variable *y* that depends on another variable *x*, intuitively we would look for a regularity in the corresponding graph (Fig. (**6.1**)) to superimpose a line or curve on the $\mathbb{R}^2$ plane or the $\mathbb{R}^3$ space that best approximates the dispersion of the data. However, in most cases, it is not evident which line or curve best approximates the dispersion of data despite the tendency of the cluster.

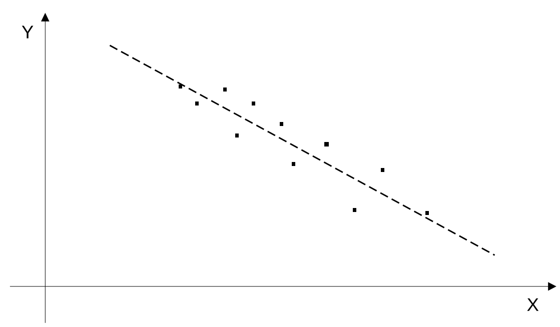


Fig. (6.1): Graphical representation of a cluster with tendency on $\mathbb{R}^2$.

In the next cluster (Fig. (**6.2**)), it is clear that any of the two lines are representative of the entire data presented. One line passing through the points on top and

Carlos Polanco

another on the points at the bottom would represent it. Another option would be a curve connecting both lines, however, that line will not be a graph of a real-valued function f (Def. (**1.2**)). We will study this restriction in Ch. (**7**).
Below, there are three frequent cases of correlation found in experimental work.

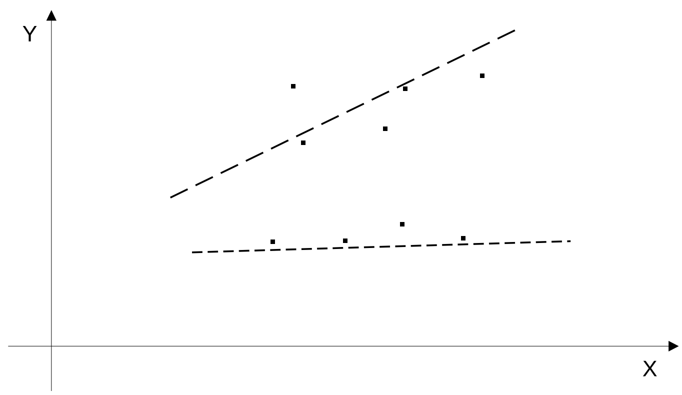


Fig. (6.2): Graphical representation of a cluster without tendency on $\mathbb{R}^2$.

Example 6.1. (i) Give a graphic example of a cluster with data tendency. (ii) Give a graphic example of a cluster without data tendency. (iii) Explain (i) and (ii).

Solution 6.1. (i) See Fig. (**6.3**).

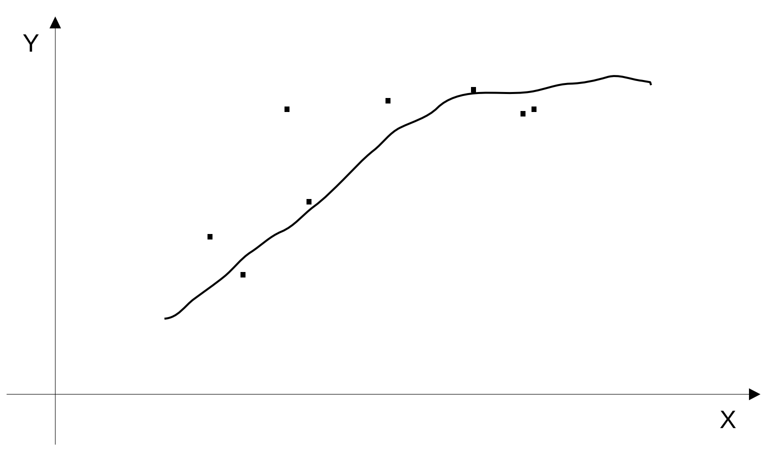


Fig. (6.3): Graphical representation of a cluster with tendency on $\mathbb{R}^2$.

(ii) See Fig. (**6.4**).

Fig. (6.4): Graphical representation of a cluster without tendency on $\mathbb{R}^2$.

(iii) In (i), it is possible to plot a curve that marks a tendency and it includes all the points. In (ii), it is not possible to determine a tendency as we cannot plot a line or a curve that represents this tendency.

6.2. Linear Correlation on $\mathbb{R}^2$

The linear correlation [13, 14] of the graphic distribution of the points (Fig. (**6.1**)) occurs when most of the points show a linear tendency i.e., when it is possible to plot a single line that represents that tendency, and the distance between the points can be determined by plotting a parallel line to the vertical axis for point *A*, as illustrated in Fig. (**6.5**).
It is not always possible to plot a line that includes all the points, sometimes one or more points will be outside this line, see Fig. (**6.2**) for the two lines marked.

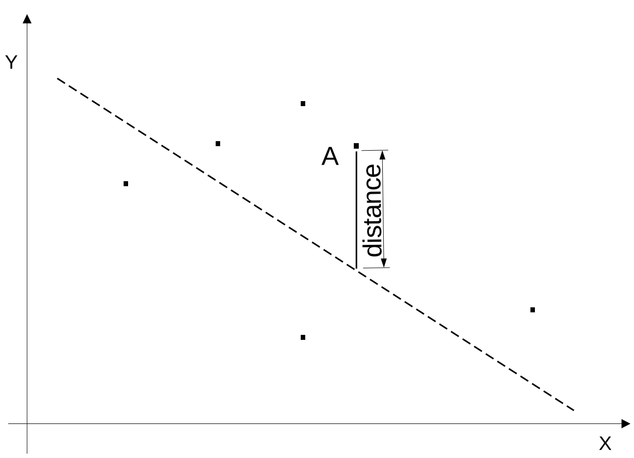


Fig. (6.5): Distance from point *A* to the line on $\mathbb{R}^2$.

CHAPTER 7

Curve Fitting

Abstract: This chapter presents an overview of curve fitting methods in Euclidean spaces, with a particular emphasis on $\mathbb{R}^2$ and $\mathbb{R}^3$. In order to represent linear and nonlinear interactions between numerous variables, a number of methodologies, including linear and nonlinear least squares methods, are being investigated. The linear relationship that exists between two variables is broken down in great detail, and a broad variety of examples are provided to show how curve fitting methods can be utilized to build models that are an accurate representation of data sets. In addition to this, the linear relationship that exists between the three variables under consideration is dissected, and detailed strategies for dealing with this scenario are discussed. Curve fitting methods are useful for exploring and evaluating data in Euclidean spaces, as shown by the results and examples shown below, which demonstrate the utility and versatility of these methods.

Keywords: $\mathbb{R}^2$ space, $\mathbb{R}^3$ space, curve fitting, least squares method, linear least squares method, linear relationship between multiple variables, linear relationship between three variables, linear relationship between two variables, nonlinear least squares method, nonlinear relationship between multiple variables, nonlinear relationship between two variables.

7.1. Introduction

In this chapter, we will delve into the fascinating world of curve fitting and least squares methods. Exploring the dimensions of space, from the two-dimensional plane to three-dimensional space, we will delve into the mathematical foundations that allow us to model and understand the relationships between variables.

We will begin our journey in the two-dimensional space, $\mathbb{R}^2$, where variables are represented as points on a Cartesian plane. Through curve fitting, we will learn to find the best approximation of an underlying relationship between two variables. To achieve this, we will employ the method of least squares, which aims to minimize the sum of squared errors between the data and the fitted curve. We will explore both linear and nonlinear relationships between two variables and learn how to determine which type of relationship best describes our data.

Moving on to three-dimensional space, $\mathbb{R}^3$, we will face a new challenge: analyzing relationships between three variables. We will investigate how least squares techniques can be applied to model linear and nonlinear relationships between three variables, enabling us to understand and predict complex phenomena in the real world.

As we delve into the various methods of curve fitting, we will focus on both linear and nonlinear relationships. We will examine in detail the linear least squares method, which is based on the assumption of a linear relationship between the variables. We will analyze how it can be used to fit data and obtain a straight line that best fits it.

Furthermore, we will venture into the realm of nonlinear relationships, where we will discover that least squares methods can also be successfully applied. We will explore the intricate nature of relationships between multiple variables and how nonlinear models can capture these relationships more accurately.

Carlos Polanco

This chapter explores key concepts of curve fitting and least squares methods within the dimensions of $\mathbb{R}^2$ and $\mathbb{R}^3$ space. From linear to nonlinear relationships, we will immerse ourselves in the world of mathematical modeling and learn to employ these powerful tools to better understand and predict phenomena in our environment.

7.2. The Method of Least Squares

The Least Squares Method is a type of mathematical regression analysis that is used to discover the polynomial function that best fits a collection of data by comparing the data with all possible functions. Every single point in the data shows the relationship between a set of known independent variables called x_i and a set of unknown values for the dependent variable called Y. The procedure reduces, to the greatest extent possible, the sum of the squares of the difference between the computed and estimated values. This value is referred to as the **distance**, and it is calculated by comparing each data point to the polynomial function Y.
The Least Squares Method is classified as Linear Least Squares Method if the relationship between variables is linear or Nonlinear Least Squares Method if the relationship between variables is nonlinear.

7.3. Linear Least Squares Method

The following applications are based on the linearity between variables and, therefore, it is necessary to verify this exists between all the variables involved. See Linear Correlation Coefficient Sect. (**6.5.1**).

7.3.1. Linear relationship between two variables

The Linear Least Squares Method on the $\mathbb{R}^2$ space defines the coefficients a_0 and a_1 of the general equation in the $\mathbb{R}^2$ space (Eq. (**7.1**)), for n number of points of the form $(x_1, Y) \in \mathbb{R}^2$. In this sense, this function relates variable x_1 to variable Y.

$$Y(x_1) = a_0 + a_1 x_1 \tag{7.1}$$

To use this method, we have to verify that variables x_1 and Y are linearly related, so it is advisable to calculate the Linear Correlation Coefficient (Sect. (**6.5.1**)).
Then, we build two normal equations from the polynomial function $Y(x_1) = a_0 + a_1 x_1$. Note that the first equation of Eq. (**7.2**) is the addition of the n points of data on $Y(x_1) = a_0 + a_1 x_1$, and the second equation of Eq. (**7.2**) is calculated with multiplying each term of the first equation with the independent variable x_1.

$$\begin{aligned} \sum_{i=1}^{n} Y &= a_0 n \quad + a_1 \sum_{i=1}^{n} x_1 \\ \sum_{i=1}^{n} Y x_1 &= a_0 \sum_{i=1}^{n} x_1 + a_1 \sum_{i=1}^{n} x_1^2 \end{aligned} \tag{7.2}$$

Remark 7.1. A normal equation is one whose coefficients a_i cannot be simultaneously zero.

Note that the system (Eq. (**7.2**)) for coefficients a_i can be solved using the Cramer's rule [15] since the sums are known, as they are formed with the points of data and its determinant is not zero.

Finaly, the coefficients obtained are replaced in the general equation $Y(x_1) = a_0 + a_1x_1$, which is the best approximation of data.

Example 7.1. Consider the points in Table (**7.1**) that relate the age of some people (x_1) to their blood pressure (Y). With this approach, blood pressure is in function of age, *i.e.* arterial pressure (age). (i) Explain why the behaviour is linear. (ii) Give the graphical representation. (iii) Obtain the n values. (iv) Once the corresponding sums are calculated replace them in Eq. (**7.2**). (v) Using Cramer's rule [15], determine a_0 and a_1. (vi) Replace them in $Y(x_1) = a_0 + a_1x_1$. (vii) Overlay graph (vi) in (ii).

Table 7.1: Age vs blood pressure.

Age	20	40	60	80	100
Blood pressure	70	75	80	85	90

Solution 7.1. (i) The values of blood pressure have a linear numerical progression, i.e., $n+5$.
(ii) See Fig. (**7.1**).

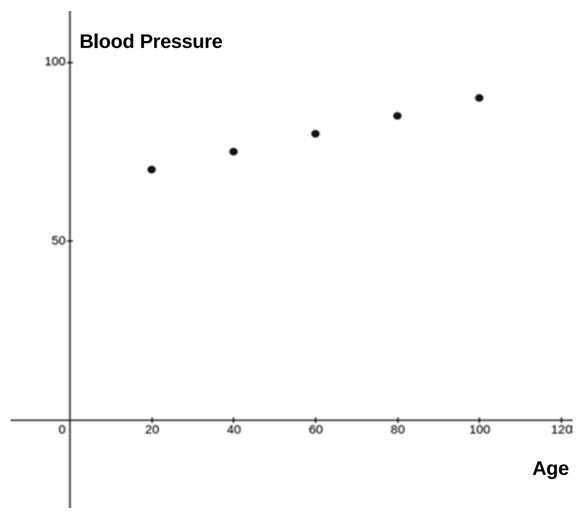


Fig. (7.1): Graphical representation of the data cluster.

(iii) See Table (**7.2**).

CHAPTER 8

Multivariate Analysis of Variance

Abstract: This chapter examines the analysis of variance (ANOVA) and Fisher statistical tests as fundamental research methods. Introduction to ANOVA and its application to the investigation of significant differences between data groups. Subsequently, we examine the application of Fisher tests in various contexts, including one-factor, two-factor, and even three-factor analyses. In addition, we investigate the design and experimental application of Latin squares. We conclude by discussing multivariate analysis of variance (MANOVA) and emphasizing its usefulness for investigating multiple response variables. This chapter concludes with a comprehensive overview of the previously mentioned statistical techniques and their applicability to scientific research.

Keywords: Analysis of variance, ANOVA, fisher statistical test, fisher test in rour-factor, fisher test in one-factor, fisher test in two-factors, fisher test in three-factors, latin squares, MANOVA, multivariate analysis of variance.

8.1. Introduction

The analysis of variance (ANOVA) [17, 18] is a standard statistical method that is used to investigate the existence of significant differences between groups. In this chapter, we will discuss ANOVA and different variants of the well-known Fisher statistical test as well as the applications of each of these variants.

In order to get started, we will get familiar with the fundamentals of ANOVA and how it connects to the concept of variance within the framework of group comparisons. We will discuss the prerequisite assumptions that must be met in order to carry out an appropriate ANOVA, as well as the many ways in which the results might be interpreted.

As we go further, we will investigate the well-known Fisher test, which is sometimes referred to as the F-test and is the basis for the analysis of variance (ANOVA). In order to evaluate the relevance of our findings in terms of statistics, we will study how to compute the F-test and how to interpret its critical value.

Moving forward, we are going to investigate several distinct iterations of the Fisher test. In this section, we will examine how to apply the Fisher test to designs with one factor, two factors, and three factors in order to investigate the interactions between independent variables and the impact these interactions have on the variable that is of interest to us.

In addition, we will investigate the concept of Latin squares, which is an experimental design that allows for the reduction of potentially confounding effects and the execution of more accurate comparisons across numerous treatments.

In addition, we will discuss the Multivariate Analysis of Variance (MANOVA), which broadens the scope of the ANOVA methodology to include the investigation of numerous dependent variables all at once. We will study how the multivariate analysis of variance (MANOVA) allows us to simultaneously assess numerous dependent variables and decide whether or not there are significant differences between the groups.

In a nutshell, this chapter will take us deep into the intriguing field of statistical analysis, namely the analysis of variance and the Fisher tests. We will investigate

Carlos Polanco

how to use these tools to conduct rigorous research and arrive at results that are statistically significant through the use of actual examples and applications developed in R. Get ready to gain an in-depth understanding of these key statistical methods through hands-on experience.

8.2. Fisher Statistical Test

To calculate the probability p, in a general way, from a contingency table, consider Table (**8.1**).

Table 8.1: Distribution of subjects with/without COVID-19 symptoms who took the PCR test. (+) positive. (−) negative.

		Row factors	
		+	−
Column factors	+	a	b
	−	c	d

The p between factors is determined by Eq. (**8.1**), where $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$. Thus, $\max\{0, a-d\} \leq a \leq \min\ \{a+b\}$, so the probability sought s$P(a=t)$ is the sum of the values calculated, *i.e.* $\sum_{i=0}^{n} P(a \geq a_i)$ such that each of these probabilities is less than the probability sought, and therefore, the expression $P(a=t) = \sum_{i=0}^{n} P(a \geq a_i)$.

$$P(a=t) = \sum_{i=1}^{n} \frac{\binom{a+c}{a}\binom{b+d}{b}}{\binom{a+b+c+d}{a+b}} \tag{8.1}$$

Example 8.1. Consider the contingency table (Table (**8.2**)) taken from Ex. (**4.2**). In this table, we exemplify the relationship between the proteins expressing SARS-CoV-2 (detected by the Polymerase Chain Reaction Test) and the symptoms of COVID-19, when $\alpha = 0.05$. (i) Determine the extreme occurrence for (i). (ii) From (i), calculate the probability of $P(a=t)$ with Eq. (**8.1**) for Tables (**8.2**) and (**8.3**). (iii) Calculate the occurrence probability of (i) or the extreme. (iv) Discuss.

Table 8.2: Distribution of subjects with/without COVID-19 symptoms who took the PCR test. (+) positive. (−) negative.

		Symptoms and Signs of COVID-19 Disease	
		+	−
Polymerase Chain Reaction Test	+	1	6
	−	4	1

Solution 8.1. (i)

Table 8.3: Extreme distribution of subjects with/without COVID-19 symptoms who took the PCR test. (+) positive. (−) negative.

		Symptoms and Signs of COVID-19 Disease	
		+	−
Polymerase Chain Reaction Test	+	0	7
	−	5	0

(ii)

$$P(a=1) = \frac{\binom{1+4}{1}\binom{6+1}{6}}{\binom{1+6+4+1}{1+6}} = 0.04399 \tag{8.2}$$

$$P(a=0) = \frac{\binom{0+5}{1}\binom{7+0}{7}}{\binom{0+7+5+0}{0+7}} = 0.0.00126 \tag{8.3}$$

(iii) $P_i = 0.04399 + 0.00126 = 0.04525$
(iv) If $P_i < \alpha$, there is a bias. We verify that $P_i = 0.04525 < \alpha = 0.05$ then, the bias exists.

(ii) Particularly, the Fisher distribution is used to demonstrate the variation between populations. Specifically, the F distribution is used when studying the ratio of the variances σ_1^2 y σ_2^2 of two normally distributed populations. For this purpose, a sample is taken from each population with sizes n_1 and n_2, so there are two degrees of freedom $n_1 - 1$ and $n_2 - 1$, respectively. Then, the variances of both populations are related in a quotient with the variances S_1^2 y S_2^2 of these samples to determine the distribution of F (Eq. (**8.4**)).

$$F = \frac{\frac{n_1 S_1^2}{\sigma_1^2 (n_1-1)}}{\frac{n_2 S_2^2}{\sigma_2^2 (n_2-1)}} \tag{8.4}$$

Note that F determines the difference between the variance of the populations and the graph corresponds to the function (Eq. (**8.5**)) for $n_1 - 1$ and $n_2 - 1$ degrees of freedom. See Fig. (**8.1**).

$$f(F) \approx \frac{F^{\frac{n_1-1}{2}-1}}{(n_1-1)F + (n_2-1)^{\frac{n_1+n_2-2}{2}}} \tag{8.5}$$

Part IV

USING A STOCHASTIC MODEL

The Discrete Markov Chain Models and the Hidden Markov Models are discussed in the part, both of which are cutting-edge areas of stochastic mathematical modeling. Case studies reviewing the approaches used to illustrate both concepts are provided. Given the intricacy of these two methods, current tools for stochastic modeling are stated in straightforward terms.

CHAPTER 9

Discrete-Time Markov Chains

Abstract: In this chapter, we look at how Monte Carlo simulations and the Markov chain theory can be used to analyze urban transportation problems. Vectors describing the starting and final states of a public transportation network are introduced as fundamental notions. The transition probability matrix and the stochastic matrix are investigated as potential tools for modeling the dynamic evolution of urban mobility using discrete-time Markov Chains. The features of Markov Chains, as revealed by their eigenvalues and eigenvectors, are examined. The Markov Chain Monte Carlo technique for statistical sampling and analysis of urban mobility issues is also discussed. The methodologies' potential utility in urban transportation planning and decision-making is emphasized. Understanding and addressing the difficulties of urban mobility are greatly aided by the theoretical and conceptual groundwork laid out in this chapter.

Keywords: Discrete-time Markov chain, eigenvalues, eigenvectors, Markov chain Monte Carlo, regular matrix, states, stochastic matrix, transition probabilitiesr,. transition probability matrix, urban mobility issues, vector in the initial state, vector in a steady state.

9.1. Introduction

In this chapter, we will explore a novel and powerful approach to tackle the complexities of urban mobility issues. Urban mobility is a highly relevant topic in today's world, as cities are undergoing rapid growth and facing significant challenges in ensuring efficient flow of people and resources.

To better understand these challenges, we will employ tools from probability theory and Markov chain theory. These theories provide a solid mathematical framework for analyzing and modeling systems that change state over time.

We will embark on our journey by introducing two fundamental concepts: the vector in the initial state and the vector in a steady state. These vectors allow us to characterize the state of a system at a given moment and in a long-term stable state, respectively. We will explore how these concepts are relevant to understanding urban mobility and how they can be used to make informed decisions.

Next, we will delve into the study of discrete-time Markov chains, which are mathematical models that describe the evolution of a system in discrete states over time. We will analyze how transitions between states are determined by transition probabilities, which will be represented in a transition probability matrix.

In our analysis, we will also examine the concepts of eigenvalues and eigenvectors. These elements allow us to understand how a system behaves with respect to state transitions and which states are more relevant for analysis and decision-making.

Additionally, we will explore the Markov chain Monte Carlo method, a powerful significant technique for simulating and analyzing complex systems based on Markov chains. This technique will enable us to address challenges and uncertainties associated with urban mobility more effectively.

Ultimately, our goal will be to utilize these concepts and tools to analyze urban mobility issues. We will examine stochastic and regular matrices, which play a key role

Carlos Polanco

in representing urban mobility systems. We will also study transition probabilities and how they influence the dynamics and flows of transportation systems.
As we progress through this chapter, we will discover how these tools and concepts allow us to gain a deeper understanding of urban mobility issues and how we can utilize this knowledge to address and resolve the challenges faced by cities in the 21^{st} century.

9.2. Process of Discrete-Time Markov Chains

The process of **Discrete-Time Markov Chains** [19, 20], has multiple variables evolving randomly in a discrete time i.e., they transit from one state to another in a single time unit. This model is particularly designed to accept multiple variables without increasing the difficulty of the model. Its methodology is not complicated and it does not have a high degree of abstraction, on the other hand, its results are quite reliable.

Definition 9.1. The **Discrete-Time Markov chain model** is a transition probabilities matrix P and vector of initial state u_0 that occurs at random. Its **states** take place in discrete time $t \in \mathbb{N}$ that satisfy $u_n = u_0P^n$, where the vector in a steady state u_n depends **only** on the previous vector u_{n-1}.

The **transition probability matrix** P^n in step n is the **discrete time** $t = n$, and the **vector in the initial state** u_0 is u_0P^n.

9.3. Markov Chain Model

A Markov Chain Model is a stochastic model that operates from a transition probability matrix, where the conditional probabilities are defined to transit from one state i to another state j. This matrix is multiplied by an vector in the initial state, which defines the probabilities of the present moment between states. From these two mathematical elements (matrix and vector), an iterated multiplication is carried out between them to get the vector of final conditions, which represents the future state the vector of initial probabilities.
The convenience of using this stochastic model is that multiple Markov Chain Models can be added to the system to represent different factors or parameters, without increasing the complexity the phenomenon simulated.
Given the transition probability matrix in the **Markov chain model**, if the vector in the initial state u_0 is set for the time t_0, the **vector in a steady state** u_1P will be the vector u for the discrete time $t = 1$, and the vector u_2P will be the vector in a steady state u_2 for $t = 2$, and so on. Therefore, $u_f = u_0P^n$ where the vector in a steady state u_n will **only** depends on the previous vector u_{n-1}.
The **transition probability matrix** P^n on state n is the **discrete time** $t = n$, and the **vector in a steady state** u_f is uP^n. So, the iteration in n states enables the transformation of the vector in the initial state into the final state vector.

Example 9.1. A shopper frequents either store A or store B on a daily basis. However, if he travels to store B, there is an equal chance that he will go to either store

A or store B the next day. He never makes a second visit to store A on consecutive days. Find the matrix that represents the transition probability.

Solution 9.1. The transition probability matrix is Eq. (**9.1**).

$$P = \begin{pmatrix} P(a_1|a_1) & P(a_2|a_1) \\ P(a_1|a_2) & P(a_2|a_2) \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} \tag{9.1}$$

Example 9.2. With Ex. (**9.1**) and the vector in the initial state $u_0 = (0.5, 0.5)$, determine: (i) u_i for $i = 1,2,3,4,5$). (ii) with $u_f = u_0 P^n$, determine u_i for $i = 1,2,3,4,5$). (iii) discuss the results.

Solution 9.2. (i) See Eq. (**9.2**)-(**9.6**).

$$u_1 = u_0 P = (0.500, 0.500) \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} = (0.250, 0.750) \tag{9.2}$$

$$u_2 = u_1 P = (0.250, 0.750) \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} = (0.375, 0.625) \tag{9.3}$$

$$u_3 = u_2 P = (0.375, 0.625) \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} = (0.312, 0.688) \tag{9.4}$$

$$u_4 = u_3 P = (0.312, 0.688) \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} = (0.344, 0.656) \tag{9.5}$$

$$u_5 = u_4 P = (0.344, 0.656) \begin{pmatrix} 0 & 1 \\ 0.5 & 0.5 \end{pmatrix} = (0.328, 0.672) \tag{9.6}$$

(ii) See Eq. (**9.7**)-(**9.11**).

$$u_1 = u_0 P^1 = (0.500, 0.500) \begin{pmatrix} 0 & 1 \\ 0.500 & 0.500 \end{pmatrix} = (0.250, 0.750) \tag{9.7}$$

$$u_2 = u_0 P^2 = (0.500, 0.500) \begin{pmatrix} 0.500 & 0.500 \\ 0.250 & 0.750 \end{pmatrix} = (0.375, 0.625) \tag{9.8}$$

$$u_3 = u_0 P^3 = (0.500, 0.500) \begin{pmatrix} 0.250 & 0.750 \\ 0.375 & 0.625 \end{pmatrix} = (0.312, 0.688) \tag{9.9}$$

$$u_4 = u_0 P^4 = (0.500, 0.500) \begin{pmatrix} 0.375 & 0.625 \\ 0.312 & 0.688 \end{pmatrix} = (0.344, 0.656) \tag{9.10}$$

$$u_5 = u_0 P^5 = (0.500, 0.500) \begin{pmatrix} 0.312 & 0.688 \\ 0.343 & 0.657 \end{pmatrix} = (0.328, 0.672) \tag{9.11}$$

SOLUTIONS

Solutions for Chapter 1

Solution 1.1.

$$f(2) = 2^2 + 3(2) + 2 = 12$$

Solution 1.2. The function is even because

$$g(-x) = -3(-x)^2 = -3x^2 = g(x)$$

Solution 1.3. The function is reflected about the x-axis, vertically stretched by a factor of 2, shifted 1 unit to the right, and shifted 3 units upward.

Solution 1.4.

$$j'(x) = (x+2)^2 + 3$$

Solution 1.5.

$$k'(x) = 4\left(\frac{x}{2}\right)^2$$

Solution 1.6. The graph is the upper half of a parabola opening to the right with its vertex at the origin.

Solution 1.7. The roots are

$$x = 0, x = 1, \text{ and } x = 3$$

Solution 1.8. The graph is a line with positive slope for $x < 0$ and a line with negative slope and a y-intercept of 2 for $x \geq 0$. Both meet at the point (0,2).

Solution 1.9. From $2x - 5 = 7$, we get $x = 6$.
From $2x - 5 = -7$, we get $x = -1$.

Carlos Polanco

Solutions for Chapter 2

Solution 2.1.

$$f(x) = x(x^2 - 3x + 2)$$
$$f(x) = x(x-1)(x-2)$$
$$\text{The roots are: } x = 0, x = 1, \text{ and } x = 2.$$

Solution 2.2.

$$\begin{aligned} 2x - 5 &= 7 && \Rightarrow x = 6 \\ 2x - 5 &= -7 && \Rightarrow x = -1. \end{aligned}$$

Solution 2.3.

$$x = \frac{\pi}{3} \text{ and } x = \frac{2\pi}{3}.$$

Solution 2.4.

$$\begin{aligned} f(2,3) &= 2^2 + 3^2 \\ &= 4 + 9 \\ &= 13. \end{aligned}$$

Solution 2.5.

$$\begin{aligned} \mathbf{r}(\pi/4) &= \begin{bmatrix} \cos(\pi/4) \\ \sin(\pi/4) \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}. \end{aligned}$$

Solutions for Chapter 3

Solution 3.1.

$$P(\text{odd number}) = \frac{3}{6} = \frac{1}{2}.$$

Solution 3.2.

$$P(\text{king}|\text{red}) = \frac{2}{26} = \frac{1}{13}.$$

Solution 3.3.

$$P(\text{defective}) = P(A) \times P(\text{defective}|A) + P(B) \times P(\text{defective}|B) = 0.6 \times 0.02 + 0.4$$

$$\times 0.03 = 0.024 \text{ or } 2.4\%.$$

Solution 3.4.

$$P(A|\text{defective}) = \frac{P(A) \times P(\text{defective}|A)}{P(\text{defective})} = \frac{0.6 \times 0.02}{0.024} = 0.5 \text{ or } 50\%.$$

Solution 3.5.

$$P(X = x) = p(x).$$

Solution 3.6. Using normal distribution tables or software, we find that

$$P(X < 70) \approx 0.9772.$$

Solution 3.7. Using the binomial formula:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$P(3 \text{ heads}) = \binom{5}{3} \times 0.5^3 \times 0.5^2 = 0.3125.$$

Solution 3.8. Using the Poisson formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

$$P(7 \text{ vehicles}) = \frac{10^7 \times e^{-10}}{7!} \approx 0.09008.$$

APPENDIX

Computational Programs

Abstract: This chapter presents a collection of Fortran-95 computer programs that address a variety of matrix computation and least squares methods-related duties. The programs include the calculation of $u = uPn$ for matrices with dimensions 3×3, 4×4, and 5×5, as well as the calculation of row vectors of in initial conditions and programs for solving problems using the least squares methods I and II. In addition, a program for urban mobility is introduced. These programs provide effective instruments for analyzing and solving complex numerical problems, allowing users to obtain precise and efficient results for practical applications. Emphasis is placed on the Fortran-95 programming language, which is renowned for its efficiency and capacity to optimally manage numerical computations.

Keywords: Calculation of $u = uP^n$ over 3×3 matrix program, calculation of $u = uP^n$ over 4×4 matrix program, calculation of $u = uP^n$ over 5×5 matrix program, calculation of row-vector of initial conditions program, FORTRAN-95, least squares method I program, least squares method II program, urban mobility program program.

A.1. Introduction

In this chapter, we will introduce a collection of Fortran-95 programs designed to solve a variety of mathematical and data analysis problems. Particular emphasis will be placed on programs associated with matrix power calculation and least squares fitting techniques.

In the first segment, we will examine a program that uses the formula $u = uPn$ to calculate the n-th power of a 3×3 matrix. This program accepts a matrix P and an integer n from the user, and computes the resulting matrix u. We will demonstrate how this program can be used to solve matrix calculation problems in a variety of contexts.

In the following section, we will present a similar program that has been modified to operate with 4×4 matrices. Matrix calculations are essential for modeling and solving complex problems in fields, such as engineering, physics, and economics. We will delve into the algorithm's implementation and discuss its potential implications in these fields.

Next, we will introduce a program designed to calculate matrix powers for 5×5 matrices in the section that follows. We will discuss the complexities of working with larger matrices and how this program can be used to solve more complex problems, such as simulating dynamic systems and analyzing complex networks.

In the next section, we will present a program dedicated to calculating initial condition row vectors. This program allows the user to input a series of data points, and then calculates a vector representing the initial conditions for a given mathematical model. We will discuss the significance of initial conditions in solving differential equations and investigate how this program can be a valuable resource for scientific inquiry.

In the final two sections of this chapter, we will discuss the programs that employ two distinct methodologies. The first program will fit a linear function to a set of

Carlos Polanco

data using the least squares method I. We will investigate the fitting algorithm and evaluate the accuracy of the obtained results using real-world examples.
In the final section, we will demonstrate a program that employs the least squares method II to fit a nonlinear function to a set of data. We will discuss the benefits and difficulties of this more complex method and illustrate its utility with numerical examples.
In conclusion, this chapter examines a variety of Fortran-95 programs that address various mathematical and data analysis issues. From matrix calculations to least-squares fitting techniques, these programs provide potent instruments for solving complex problems in a variety of academic disciplines. We hope that readers will acquire a solid understanding of these programs and be able to apply them to their own research and projects through the use of examples and detailed explanations.

A.2. Least Squares Method I Program

```
[language=octave]

!      Author: Carlos Polanco
!      Date: July, 2022.
!
      IMPLICIT none
      REAL x, y, i, z1
      REAL x1, x2, x12, x13, x14, x22, x23, x24
      REAL x1x2, x12x2, x22x1, x12x22, z, zx1, zx2
      REAL zx12, zx22, zx1x2, x1x22, x13x2, x1x23


      x1 = 0.0
      x2 = 0.0
      z1 = 0.0
      x12 = 0.0
      x13 = 0.0
      x14 = 0.0
      x22 = 0.0
      x23 = 0.0
      x24 = 0.0
      x1x2 = 0.0
      x12x2 = 0.0
      x22x1 = 0.0
      x12x22 = 0.0
      zx12 = 0.0
      zx22 = 0.0
      zx1 = 0.0
      zx2 = 0.0
      z = 0.0
      zx1x2 = 0.0
      x1x22 = 0.0
```

```
      x13x2 = 0.0
      x1x23 = 0.0
      i = 1.


      x = 1.1
      y = 2.3

   60 z = 1 + 1.3 * x + 2.4 * y + .5 *
      x**2 + 0.7 * x * y + 3.1 * y**2
      x1 = x1 + x
      x2 = x2 + y
      z1 = z1 + z
      x12 = x12 + x**2
      x13 = x13 + x**3
      x14 = x14 + x**4
      x22 = x22 + y**2
      x23 = x23 + y**3
      x24 = x24 + y**4
      x1x2 = x1x2 + x * y
      x12x2 = x12x2 + y * x**2
      x22x1 = x22x1 + y * x**2
      x12x22 = x12x22 + (x**2) * (y**2)
      x1x22 = x1x22 + x * (y**2)
      x13x2 = x13x2 + (x**3) * y
      x1x23 = x1x23 + x * (y**3)

      zx12 = zx12 + z * x**2
      zx22 = zx22 + z * y**2
      zx1 = zx1 + z * x
      zx2 = zx2 + z * y
      zx1x2 = zx1x2 + z * x * y


      x = x + .22
      y = y + .14

      PRINT *, "Paso:", i, z

      i = i + 1.
      IF (x.le.2.) goto 60
      PRINT *, "x1", x1
      PRINT *, "x2", x2
      PRINT *, "x12", x12
      PRINT *, "x13", x13
      PRINT *, "x14", x14
      PRINT *, "x22", x22
      PRINT *, "x23", x23
      PRINT *, "x24", x24
      PRINT *, "x1x2", x1x2
      PRINT *, "x12x2", x12x2
```

REFERENCES

[1] "Desmons inc." 2021, https://www.desmos.com/calculator?lang=es.
[2] C. Polanco, *Advanced Calculus: Fundamentals of Mathematics*. Bentham Science Publishers, -Sharjah, UAE.
[3] G. Grimmett and D. Stirzaker, *Probability and Random Processes*. Oxford University Press, 2001.
[4] H. Stark and J. W. Woods, *Probability and Random Processes with Applications to Signal Processing*. Pearson, 2002.
[5] S. Ross, *A First Course in Probability*. Pearson, 2019.
[6] C. M. Bishop, *Pattern Recognition and Machine Learning*. Springer, 2006.
[7] R. Radner and D. Radner, *Decision Theory: Principles and Approaches*. Springer, 2018.
[8] "Wikipedia. sensitivity/specificity:graphical illustration," 2021, https://en.wikipedia.org/wiki/Sensitivity˙and˙specificity.
[9] "Wikipedia. sensitivity/specificity:related calculations," 2021, https://en.wikipedia.org/wiki/Sensitivity˙and˙specificity.
[10] J. D. Gibbons and S. Chakraborti, *Nonparametric Statistical Inference*. Chapman and Hall/CRC, 2011.
[11] J. S. Croucher, *Nonparametric Statistics: A Step-by-Step Approach*. Wiley-Blackwell, 2014.
[12] S. Siegel, *Non parametric statistics for the behavioral sciences.* McGraw-Book Company, New York U.S.A., 1970.
[13] D. C. Montgomery, E. A. Peck, and G. G. Vining, *Introduction to Linear Regression Analysis*. Wiley, 2012.
[14] D. L. Siegel, *Correlation and Dependency*. CRC Press, 2014.
[15] "Cramer 's rule," 2022, http://es.onlinemschool.com/math/assistance/equation /kramer/.
[16] "Multivariate quadratic regression," 2021, https://math.stackexchange.com/questions/3155866/multivariate-quadratic-regression.
[17] R. A. Johnson and D. W. Wichern, *Applied Multivariate Statistical Analysis*. Pearson, 2007.
[18] B. F. Manly, *Multivariate Statistical Methods: A Primer, Fourth Edition*. Chapman and Hall/CRC, 2005.
[19] U. of Bath, "Discrete time markov chains," 2019, https://people.bath.ac.uk/ak257/36/Markov.pdf.
[20] S. M. Ross, *Introduction to Probability Models*. Academic Press, 2019.
[21] M. Goldman, "Markov chains - stationary distributions," 2008, https://www.stat.berkeley.edu/m̃goldman/Section0220.pdf.
[22] J. Lambers, "Fixed points for functions of several variables," 2009, https://www.math.usm.edu/lambers/mat461/spr10/lecture22.pdf.
[23] E. Cheever, "Eigenvalues and eigenvectors," 2021, https://lpsa.swarthmore.edu/MtrxVibe/EigMat/MatrixEigen.html.
[24] J. Navas-Urena, "Capítulo 7 modelos matriciales discretos," 2021, http://matema.ujaen.es/jnavas/web˙modelos/pdf˙mmb08˙09/modelos%20mat riciales.pdf.
[25] C. Polanco, *Markov Chain Models: Theory and Cases*. Bentham Science Publishers, -Sharjah, UAE.

Carlos Polanco

SUBJECT INDEX

A

B

C

Carlos Polanco

D

E

F

G

H

I

L

M

N

P

Q

R

S

T

U

V

Carlos Polanco

Dr. Carlos Polanco is currently Assistant Professor in the Department of Mathematics at the Universidad Nacional Autónoma de México (UNAM) since 2006. He is also a postdoctoral researcher and Head of the Department of Electromechanical Instrumentation at the Instituto Nacional de Cardiología Ignacio Chávez since 2021.

Dr. Polanco focuses his research on the design of mathematical-computational models relating to Structural Proteomics and Mathematical Epidemiology using techniques of high-performance computing, clustering and microchips solutions.

He is a mathematician and had received his PhD suma cum laude from the UNAM.