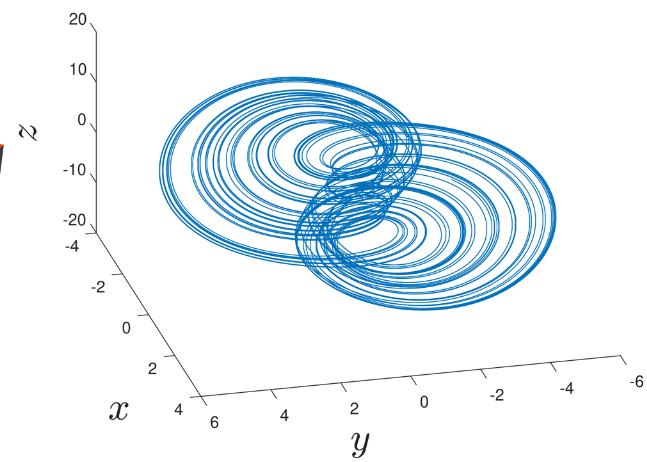
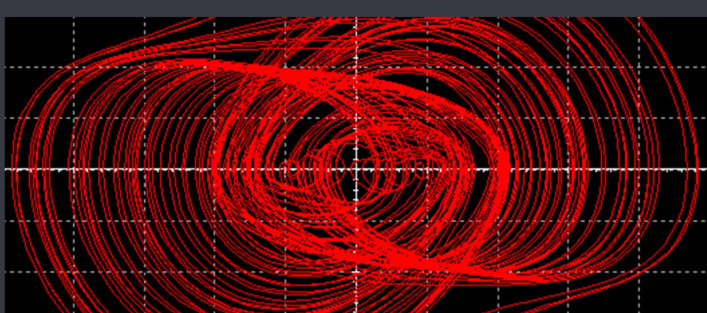
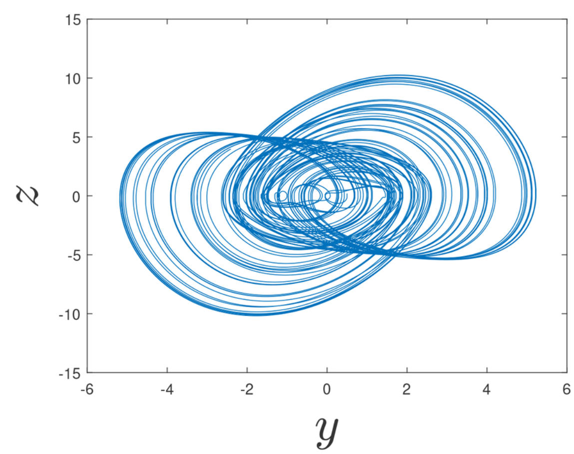
FRACTIONAL CALCULUS: NEW APPLICATIONS IN UNDERSTANDING NONLINEAR PHENOMENA







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Fractional Calculus: New Applications in Understanding Nonlinear Phenomena

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Current Developments in Mathematical Sciences

Volume # 3

Fractional Calculus: New Applications in Understanding Nonlinear Phenomena

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CONTENTS

FOREWORD

In the past few decades, fractional derivatives and integrals have been recognized as powerful modelling and simulation tools for engineering, physics, economy and other application areas. Many physical laws are expressed more accurately in terms of differential equations of arbitrary order. The fractional derivatives and integrals and their potential uses have gained a great importance, mainly since they have become powerful instruments with more accurate, efficient and successful results in mathematical modelling of several complex phenomena in numerous seemingly diverse and widespread fields of science, especially engineering, finance and biology. As the fractional dynamical systems grow, mature and develop, it is very prominent to focus on the most promising novel directions that were worked out based on the novel methods and schemes handed over recently in the field.

The key objective of this book is to focus on recent advancements and future challenges on the basic foundation and applications of the fractional derivatives and integrals in dynamical systems.

This edited book received a number of submissions, out of which 10 high-quality chapters were accepted. The chapters of this book have a large variety of interesting and relevant subjects, namely, fractional partial differential equations, chaotic systems and control, heat conduction, numerical algorithms, complexity and fractional calculus with power law, exponential decay law and Mittag-Leffler non-singular kernel.

We congratulate the Editors, Dr. Mehmet Yavuz and Dr. Necati Özdemir, who were able to collect a variety of topics of relevance to the reader and we are sure that this book will be helpful to scientists doing research in different fields of fractional calculus.

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PREFACE

The investigation of fractional integrals and fractional derivatives has a long history, and they have many real-world applications because of their properties of interpolation between operators of integer order. This field has covered the classical fractional operators such as Riemann–Liouville, Weyl, Caputo, Grunwald–Letnikov, and so on. Also, especially in the last two decades, many new fractional operators have appeared, often defined using integrals with special functions in the kernel as well as their extended or multivariable forms. These have been intensively studied because they can also be useful in modelling and analysing real-world processes, because of their different properties and behaviours, which are comparable to those of the classical operators.

This book contains ten chapters in three sections. The first section, **Chaotic Systems and Control**, contains three chapters. In Chapter 1, Sene proposed a numerical procedure and its applications to a fractional-order chaotic system represented with the Caputo fractional derivative. In Chapter 2, Okundalaye *et al.* gave a new multistage optimal homotopy asymptotic method for solutions to a couple of fractional optimal control problems. In Chapter 3, Farman *et al.* studied a complex chaotic fractional-order financial system in price exponent with control and modelling.

The second part of the book, **Heat Conduction**, contains two chapters. In Chapter 4, Hristov proposed an attempt to demonstrate that the Duhamel theorem applicable for time-dependent boundary conditions (or time-dependent source terms) of heat conduction in a finite domain and the use of the Fourier method of separation of variable (superposition version) naturally leads to appearance of the Caputo–Fabrizio operators in the solution. In Chapter 5, Avcı and İskender Eroğlu considered the oscillatory heat transfer due to the Cattaneo–Hristov model on the real line modelled by a fractional-order derivative with a non-singular kernel.

The third section of the book, **Computational Methods and Their Illustrative Applications**, contains five chapters related to different types of real-life problems. In Chapter 6, Ghoreishi *et al.* applied the optimal homotopy analysis method for a nonlinear fractional-order model to HTLV-1 infection of $CD4^+$ T-cells. In Chapter 7, Durur *et al.* investigated the behavior analysis and asymptotic stability of the traveling wave solution of the Kaup-Kupershmidt equation with the conformable operator. In Chapter 8, Baishya *et al.* took into account the Caputo fractional order derivative in the mathematical analysis of a rumor-spreading model and presented interesting numerical results. In Chapter 9, Veeresha *et al.* studied a unified approach for the fractional system of equations arising in the biochemical reaction without a singular kernel. In Chapter 10, Bora *et al.* investigated the hydro-morphodynamic effects induced by a non-powered floating object navigating in an approach channel using the CFD (Computational Fluid Dynamics) process.

We are very much thankful to all the contributors to this book for their valuable and productive works. The foreword for this book has been written by Prof. Dumitru Baleanu and Prof. Jordan Hristov. We would like to express our sincere gratitude for their guidance and support.

We are extremely grateful to Mrs. Humaira Hashmi (Editorial Manager Publications) and Mrs. Fariya Zulfiqar (Manager Publications) of Bentham Science Publishers who helped us in the publication process. We are also extending our thanks to Bentham Science Publishers for publishing this book.

We wish that this book will be especially useful to scientists doing research in the field of fractional calculus and to researchers at graduate level in this field.

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CHAPTER 1

Numerical Procedure and its Applications to the Fractional-Order Chaotic System Represented with the Caputo Derivative

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Abstract: This chapter focuses on a numerical procedure and its application to a fractional-order chaotic system. The numerical scheme will discuss the Lyapunov exponents for the considered model and characterize the chaos's nature. We will also use the numerical scheme to depict the phase portraits of the proposed fractional-order chaotic system and the bifurcation maps. Note that the bifurcation maps are used to characterize the influence of the different parameters of our considered fractional model. The impact of the initial conditions and the coexisting attractors will also be analyzed. With the coexistence, the new types of attractors will be discovered for our considered model. To confirm the investigations in this chapter, the proposed model will be applied to the electrical modeling. Therefore, the circuit schematic of the considered fractional model will be implemented in real-world problems. And we notice good agreement between the theoretical results and the results obtained after Multisim simulations. The stability of the equilibrium points of the presented model will also be focused on details and will permit us to delimit the chaotic region in general.

Keywords: Attractors, Bifurcation maps, Chaotic systems, Lyapunov exponents, Stability analysis.

INTRODUCTION

This chapter focuses on chaos theory in the context of fractional calculus. There exist many real-world applications of chaos theory in modeling electrical circuits [1, 2], engineering sciences, modeling electronics phenomena and others [4, 5]. Many differential equations admitting chaotic behaviors and hyperchaotic behaviors exist in two dimensions, three dimensions, four dimensions, five

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Mehmet Yavuz & Necati Özdemir (Eds.)

dimensions, and others dimensions. Many of them are described by the integer-order derivative. This chapter will introduce the fractional operators in the modeling of a class of chaotic systems. The fractional calculus continues its expansion, with many discussions about these applications of the derivative in real-life problems. Many address the interpretations of the fractional operators, but the consensus is not unanimous. Many derivatives exist as the old operators; we have the Caputo derivative [6, 7] and the Riemann-Liouville derivative [6, 7], and many others. We also have recent fractional operators with new types of kernels as the Atangana-Baleanu derivative, the Caputo-Fabrizio derivative, conformable derivative, and others [11-17]. For recent developments of fractional calculus and fractional operators applications, the readers can look at the following papers for examples[11, 18-24].

LITERATURE REVIEW IN CHAOS

Modeling chaotic and hyperchaotic systems with fractional operators in fractional calculus was first proposed by Petras in studies [25, 26]. The literature of chaotic and hyperchaotic systems with integer derivatives and fractional derivatives is vast. In this part, we recall some of them. Petras studies various classes of fractional-order chaotic systems described by the Caputo derivative, the stability of the equilibrium points of the chaotic systems is also presented in its book [25]. An algorithm to obtain the Lyapunov exponents in the context of fractional calculus by Matlab software has been presented by Danca and his co-author [27]. Ren and co-authors presented a new chaotic system flow with the presence of a hidden attractor; this new system is known to belong to the jerk systems with no equilibrium points [28]. Rajagopal *et al.* presented the so-called chameleon fractional chaotic system [29]. Vaidyanathan and the co-authors presented a hyperchaotic system with five dimensions [2]. The authors also presented the circuit schematic of their model, and the results are represented in oscilloscopes. Pham *et al.* presented the coexistence attractors of a hidden chaotic system with no-equilibrium points [30]. The authors presented a new class of chaotic systems and developed some properties related to their presented novel model [31]. The authors developed control and synchronization of the fractional chaotic system using an active controller [32]. Diouf *et al.* proposed the phase portraits and bifurcation maps of the three-dimensional financial chaotic differential equation using a numerical scheme in the context of fractional calculus [33]. Sene used Caputo derivative to model financial model in four-dimensional space [34]. The system studied is hyperchaotic but sometimes with one positive Lyapunov exponent and sometimes with two positive Lyapunov exponents. The properties related to the Lyapunov exponents in fractional context are well detailed in this work. Sene *et al.* proposed an investigation related to chaotic and hyperchaotic systems described by the Caputo derivative [35]. They studied the proposed system the qualitative properties using the Lyapunov exponents and phase portraits. Sene analyzed the class of fractional-order chaotic system described by the Caputo derivative using bifurcation and Lyapunov exponents [36]. See also the study [37] for more investigations.

MOTIVATIONS AND NOVELTIES

Modeling fractional-order chaotic systems will be the main innovation of this chapter. The chaotic system will be represented using the Caputo derivative. To obtain the phase portrait, discretization, including the discretization of the Riemann-Liouville integral and the analytical solution, will be used. The fractional-order will generate different types of attractors, and the Lyapunov exponents will be used to classify them. The Lyapunov exponents in the context of the fractional derivatives, as proposed by Danca [27], will be illustrated. Note that one positive Lyapunov exponent means the existence of chaotic behaviors in general. The impact of the initial conditions, the coexisting attractors will be analyzed, and the variation of the proposed model's parameters will be illustrated using the bifurcation maps. The stability analysis will delimit the interval under which the chaotic behaviors exist when we use the Caputo derivative. Many other qualitative properties of the dynamic under investigations will be presented, illustrated, and discussed as possible in this chapter.

FRACTIONAL OPERATORS AND DERIVATIVES

Many operators exist in fractional calculus with and without singular kernel. In this chapter, we try to recall two of them which will be of great interest to us for our investigation. Caputo, Riemann, and Liouville propose the fractional derivatives most used in fractional calculus for decades. We defined the fractional integral before these derivatives, know as the Riemann-Liouville derivative, as the following definition.

The representation of the Riemann-Liouville integral of the function z is described in the following formula

$$\left(I^{\alpha}z\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\, z(s)ds,$$

where $\Gamma(\dots)$ defines the Gamma Euler function and and the order α is imposed to respect the condition $\alpha > 0$ [6, 7]. The fractional integral operator plays an essential role in discretization. Its discretized form is well known in the literature and can be

CHAPTER 2

A New Method of Multistage Optimal Homotopy Asymptotic Method for Solution of Fractional Optimal Control Problem

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Abstract: This paper deals with a recent approximate analytical approach of the multistage optimal homotopy asymptotic method (MOHAM) for fractional optimal control problems (FOCPs). In this paper, FOCPs are developed in terms of a conformable derivative operator (CDO) sense. It is validated that the right CDO appears naturally in the formulation even when the system dynamics are described with the left CDO only. The CDO is employed to enlarge the stability region of the dynamical systems of the optimal control problems (OCPs). The necessary and transversal conditions are achieved using a Hamiltonian technique. The results demonstrated that as the fractional-order solution derivative tends to integer-order 1, the formulations lead to integer-order system solutions. Numerical results and a comparison with the exact solution and other approximate analytical solutions in fractional order are given to validate the efficiency of the MOHAM. Some numerical examples are included to demonstrate the effectiveness and applicability of the new technique.

Keywords: Approximate analytical solution, Convergence analysis, Conformable derivative operator, Fractional calculus, Fractional Hamiltonian approach, Fractional optimal control problems.

INTRODUCTION

The global definition of an optimal control problems depends on the minimization

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Mehmet Yavuz & Necati Özdemir (Eds.)

of an objective function of the state and control inputs of the system over a set of relevant control functions. The OCPs usually emerge in diverse areas of applied science, and well-founded studies have been done in the classical derivatives dynamic systems.

A non-linear constrained OCPs can be of various types, relying on the conditions constrained on the final time and state. It can be grouped as fixed final state-fixed final time, free final state-fixed final time, fixed final state-free final time, and free final state-free final time. Most computing techniques for the solution of OCPs conveniently solved the unconstrained problem, but inequality constraints often resulted in both exact and numerical computational difficulties. In control, the state-space illustrations are an effective approach for stability analysis and OCPs formulation. Fractional calculus (FC) is the generalization of traditional calculus and has drawn the attention of several authors in the areas of applied science to described more precisely the dynamics of many systems using fractional calculus. It has been divulged in the literature that systems represented using FC give more interesting behaviour [1-5]. Also, it has been demonstrated that the substances with memory, genetic properties, heat conduction, and gas diffusion can be modeled more accurately with FC [6-9]. Many definitions of fractional-order derivative operator (FODO) can be seen [10-16]. The Riemann-Liouville (RL) derivative operator is not consistently usable for modeling physical systems because the RL solution requires unnatural initial conditions [17-19]. In contrast, the Caputo fractional derivative operator (CFDO) accepts initial conditions like the integer-order systems. Thus, CFDO is good for modeling physical systems [20-23], epidemiological analysis of COVID-19 [24], discretization method for an expanded family of distributions [25], fractional-order COVID-19 epidemic model [26], the transmission of COVID-19 dynamic system [27], fractional Burger–Fisher equations [28], time-fractional Burgers-Coupled equations [29-31], and time-fractional Fisher's equations [32, 33]. The fractional-order derivative operators have been applied to many problems in the area of OCPs. We refer the researchers who are interested in the theory and applications of FC to these books [34-39] with some papers on FOCPs [40-43] . The FOCPs of several cases have been constructed and considered using various variations of FC: Riemann-Liouvile for FOCPs [44, 45], Mittag-Leffler for FOCPs [46] Caputo for FOCPs [47], and Atangana-Baleanu for FOCPs [48]. Agrawal gave a general formulation and solution scheme for a class of FOCPs in terms of the RL [49]. Recently, fractional conservation laws for FOCPs with RL fractional derivatives (RLFDs) were studied [50], FOCPs in Caputo sense were addressed [51], one state and one control variable, and one fractional state equations [52], and Hamiltonian equations for fractional variational problems [53]. The FOCPs are OCPs in which the differential equations (DEs) governing the dynamics system exhibit at least one FODO [54]. Authors in [55], gave a pseudo-state-space-based FOCPs formulation and a solution scheme. Fixed and free final-time FOCPs are considered in the study [56], second-order necessary optimality condition for FOCPs in the Caputo sense [57], and FOCPs of an HIV-immune system in terms of the Caputo sense [58]. Recent approximate analytical methods (AAM) are: modified Adomian decomposition method for (FOCPs) [59], variational iteration method for optimal solutions FOCPs [60], conformable fractional optimal control problem of heat conduction equations using Laplace and finite Fourier sine transforms [61], spectral Galerkin approximation [62], new approximate-analytical solutions for PDEs [63], transcription methods for FOCPs [64], but the methods mentioned above lack convergence criteria and interval of convergent.

In 1992, Liao proposed the homotopy analysis method (HAM), independent of any small or large physical parameters [65]. Homotopy methods are robust mathematical tools for obtaining a solution to many non-linear problems. Contrary to all other approximate analytic methods, it gives us a convenient means to guarantee the convergence of the series solution of non-linear problems by putting in an auxiliary parameter, called the convergence-control parameter (CCP), and offers a solution to this problem where the exact solution is not available. Using this parameter, we can easily control and possibly extend the convergence region of the solution obtained; this merit makes HAM an excellent method for applied mathematicians [66]. It has been verified that the HAM solution with this CCP is a Taylor series expansion of the analytical solution at some point [67]. In 2003, the elementary concepts of HAM and some applications largely related to non-linear ODEs were described systematically by the author in the book “Beyond Perturbation” [68]. The HAM has attracted many researchers in nearly a score of nations. It has been intensely employed to resolve many non-linear problems in a scientific discipline, finance, and technology [69] for solving the non-linear problem, which was later advanced to OHAM for non-integer order [70], new fractional homotopy method for OCPs [71], optimal control of a constrained fractionally damped elastic beam [72], and comparisons of OHAM [73]. But MOHAM has never been used to solve FOCPs, which drives this research work. The present work aims to find the approximate analytical solutions for FOCPs using a new novel technique called MOHAM. Our focus in this paper is to widen the application of MOHAM to obtain accurate solution of FOCPs. We provide answers to convergence criteria for accurate optimal solutions, the convergence of series solutions, and the solution using MOHAM with an optimization technique of the Galerkin method. The merit of this paper is that CDO is employed to enlarge the stability region of the dynamical. The scope covers the limitations in approximate

CHAPTER 3

Complex Chaotic Fractional-order Finance System in Price Exponent with Control and Modeling

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Abstract: The present chapter proposes modeling of complex fractional-order chaotic financial system with control. Here, we have added critical minimum interest rate 'd' as a new parameter to get a novel stable financial model. The fractional derivatives. are taken in Caputo and Caputo-Fabrizio sense for the proposed finance system. Dynamical models in financial system with complicated behavior provide a new. perspective as result of trends and actual behavior of internal structure of the financial. system. A theoretical stabilization of the equilibria, as well as the numerical simulations, are obtained. Furthermore, with sensitivity analysis, a certain threshold estimation of the basic reproductive number has been made. Also, the stability analysis of the model, together with uniqueness of the special solutions is provided. The concept of controllability and observability for the linearized control model is used for feedback control. The solution of the proposed fractional-order model has been procured by employing different numerical techniques with comparison among the solutions. The convergence analysis is carried out for the accuracy of the applied scheme. Finally, some numerical simulations are given for three fractional-order chaotic systems to verify the efectiveness for the obtained results. The fractal, stochastic processes and prediction are used in particular mechanism of its application to the macro and micro processes.

Keywords: Complex chaotic system, Caputo derivative, Caputo-Fabrizio derivative, Dynamical control, Fixed point theorem, Fractional-order finance. system, Stability analysis.

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Mehmet Yavuz & Necati Özdemir (Eds.)

INTRODUCTION

The financial system has two classes which are called Microeconomics and Macroeconomics. Microe- conomics is an individual's study, business decision and Macroeconomics is an extensive study. Consequently, the financial systems works at particular, domestic and international level. In the financial system, money, credit and finance are used as a media of exchange. Financial and economic systems are getting harder to understand and economic growth varies from low to high financial markets. The process of eco- nomic development and growth is more complex on the basis of multiple variables. There are non-linear factors like interest rate, good's price, investment demands and stock. In Economics, mathematicians started to apply chaos theory [1, 2] during the last decades of the 20^{th} century.

The mathematical behavior of the fractional order system is studied. If input on the system is influenced according to time, only then it is called static system. If the current and past input on the system is influenced, then it is called dynamical system. To gain the desired doutput and adjust system input, a controller is used which is considered a system named as a controlled system. The controller is said to be a closed-loop controller if the controlled output on system directly depends on controller inputs otherwise, it is said to be an open-loop controller [3].

The oldest mathematical tool which provides attractive research in all kinds of fields [4] is fractional calculus. Fractional calculus plays the main role and has many benefits as compared to integer calculus that narrates the memory and hereditary characteristics of different procedures and materials [5]. Financial variables are appropriate to use fractional modelling, which narrates the actual behavior of the financial system and possesses long memories (actual behavior of the financial system which possesses long memories; stock market prices, exchanges rates and interest rates are considered to possess long memories with very influential behavior to initial prices or values and some chaotic attractors (28)). Chen proposed a fractional order financial system, which narrates the actual dynamical behavior, studied the period-doubling and identified intermittency routes to chaos [6]. Financial system is studied by the chaos control method for slide mode and feedback control [7, 8]. The numerical techniques are used to solve such complex financial systems as exact solution cannot be found easily, the most used technique to solve fractional differential equations is the GABMM [9]. Some analytical methods to solve nonlinear differential equations are VIM, HPM, ADM and homotopy analysis method [10-12].

Since in recent years, fractional calculus has been an important gadget to describe the dynamical behaviour of different physical systems [11]. In recent years, researchers have taken interest and attention to fractional calculus in different aspects under consideration for research of the said subject [13, 14]. In the last decade, derivatives and integrals of fractional orders had notable development as revealed by several monographs dedicated to it, studied differential-difference equation of fractional order [13-15].

In this chapter, to develop the system of complex nonlinear differential equations, we apply fractional parameters using the Caputo and Caputo Fabrizio derivatives method. Dynamical models of the financial system for complicated behavior are checked from a new perspective as result of trends and actual behavior of internal structure of the financial system. The stabilization of equilibrium is obtained by both theoretical analysis and simulation results. The linearized systems of controllability and observability are designed for the close loop of automatic control.

MATERIALS AND METHOD

The subject of Mathematical research is fractional calculus which is the result of integral value exponents from the traditional definition of integral calculus and derivative operations as fractional exponents [16-23].

Definition 2.1. For a function $g\colon \mathfrak{R}^{+} \to \mathfrak{R}$, then the fractional integral of order $\beta > 0$ is given by

$$I_t^{\beta}\big(g(t)\big) = \frac{1}{\gamma(\beta)}\int_0^t (t-z)^{\beta-1} g(z)dz,$$

where γ shows the Gamma function and β is the fractional order parameter.

Definition 2.2 For a function $g \in C^n$, then the Caputo derivative of order $\beta > 0$ is defined by [9]

$$_{C}D_t^{\beta}\big(g(t)\big) = I^{n-\beta}D^n g(t)\frac{1}{\gamma(n-\beta)}\int_0^t \frac{g^n(z)}{(t-z)^{\beta+n-1}}dz,$$

CHAPTER 4

The Duhamel Method in Transient Heat Conduction: A Rendezvous of Classics and Modern Fractional Calculus

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Abstract: This chapter presents an attempt to demonstrate that the Duhamel theorem applicable for time-dependent boundary conditions (or time-dependent source terms) of heat conduction in a finite domain and the use of the Fourier method of separation of variable (superposition version) naturally lead to appearance of the Caputo-Fabrizio operators in the solution. The fractional orders of the emerging series of Caputo-Fabrizio operators are directly related to the eigenvalues determined by the Fourier's method. The general expression of the solution in terms of Caputo-Fabrizio operators has been developed followed by four examples.

Keywords: Caputo-Fabrizio derivative, Duhamel theorem, Heat conduction.

INTRODUCTION

The chapter is especially devoted to the idea of demonstrating how the classical methods in analytical heat transfer (diffusion) meet the modern fractional operators. The target is the well-known Duhamel's method allowing the transient heat diffusion problem with time-dependent boundary conditions to be presented as a convolution integral (see in the sequel). The general form of the Caputo-type fractional operator can be expressed as

$$D_t^{\alpha} f(t) = \frac{M(\alpha)}{N(\alpha)} \int_0^t R(\alpha, t-\tau) \frac{df(\tau)}{d\tau} d\tau,$$

where $M(\alpha)/N(\alpha)$ is a normalization function and $R(\alpha, t-r)$ is the relaxation function (memory kernel).

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Mehmet Yavuz & Necati Özdemir (Eds.)

From this general definition, we may briefly present two general members of this group.

Caputo derivative [1]

$$ {}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha}} \frac{df(\tau)}{d\tau} d\tau, \quad 0 < \alpha \leq 1, $$

with a singular (power-law) memory kernel, $M = 1$ and $N(\alpha) = \Gamma(1-\alpha)$

and *Caputo-Fabrizio fractional operator* [2]

$$ {}^{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^t exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] \frac{df(\tau)}{d\tau} d\tau, \qquad 0 < \alpha \leq 1 $$

with a non-singular (regular) memory kernel where $N(\alpha) = 1 - \alpha$, while the function $M(\alpha)$ satisfies the conditions $M(0) = M(1) = 1$.

From this position, we try to show that many solutions to transient problems developed by Duhamel's method can be presented in terms of the Caputo-Fabrizio fractional operator by applying classical solution methods, such as the Fourier's separation of variables.

The Main Focus of this Chapter

The modern treatments in fractional calculus try to make the fractional operators useful and versatile tools for solving real-world problems. Nowadays, there are too many controversial opinions about the adequate applications of fractional operators with different memory kernels. This requires the development of examples where it is possible clearly and on the basis of well-known methods of solutions to show that the availability of the new fractional operators, especially that of Caputo-Fabrizio, is natural. The main focus of this chapter is on the basis of classical solutions available in many textbooks on heat transfer to demonstrate that by applying the Duhamel's method and the method of separation of variables, the Caputo-Fabrizio operators appears naturally: precisely, solutions as series of operators with fractional parameters depending on the eigenvalues of the auxiliary transient problem.

TRANSIENT HEAT CONDUCTION AS A PRINCIPLE PROBLEM

Let us consider the classical heat conduction problem with constant heat diffusivity

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2} \tag{1}$$

formulated in a finite domain (1-D slab), $0 \leq x \leq L$ under homogenous initial and time-dependent boundary condition at $x = 0$, namely

$$T(x,0) = 0, \quad T(0,t) = Q(t), \quad T(L,t) = 0. \tag{2}$$

The solution of this problem needs application of the Duhamel method briefly presented next.

Duhamel's Method

In one dimensional case, with zero initial conditions, which are the mandatory conditions of the Duhamel theorem [3], in a finite domain and Cartesian coordinates, the models is (1) with boundary and initial conditions (2).

The application of the Duhamel method needs a solution of an auxiliary problem where the model (1) has to be solved with a unit step change at the $x = 0$ (Dirichlet problem) ($(t) = U(t)$, $U(t) = 0$ for $t < 0$ and $U(t) = 1$ for $t \geq 0$), namely

$$T(0,t) = U(t), \quad T(L,t) = 0, \quad T(x,0) = 0$$

Therefore, we may formulate the auxiliary problem as

$$\frac{\partial R}{\partial t} = a\frac{\partial^2 R}{\partial x^2}, \quad 0 < x < L \tag{3}$$

with with boundary and initial conditions

$$R(0,t) = 1, \quad R(L,t) = 0, \quad R(x,0) = 0 \tag{4}$$

In order to solve the original problem (1)-(2) the solution can be presented in the form, within a limited time interval $0 \leq t \leq r_1$, by applying the solution of the auxiliary problem (3)-(4), in a general form

$$T(x,t) \equiv Q(0)\,R(x,t)$$

CHAPTER 5

Oscillatory Heat Transfer Due to the Cattaneo-Hristov Model on the Real Line

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Abstract: This chapter aims to discuss the analytical solutions for heat waves observed in Cattaneo-Hristov heat conduction modelled with Caputo-Fabrizio fractional derivative. This operator includes a non-singular exponential kernel and also requires physically interpretable initial conditions for its Laplace transform property. These provide significant advantages to obtain analytical solutions. Two different types of harmonic heat sources are assumed to elicit heat waves. The analytical solutions are obtained by applying Laplace transform with respect to the time variable and the exponential Fourier transform with respect to spatial coordinate. The temperature curves for varying values of the fractional parameter, angular frequency, and the velocity of the moving heat source are drawn using MATLAB.

Keywords: Caputo-Fabrizio fractional derivative, Cattaneo-Hristov heat diffusion model, Exponential fading memory, Fourier transform, Harmonic source effect, Laplace transform, Oscillatory heat transfer.

INTRODUCTION

The diffusion phenomenon describes the movement of various materials in nature, such as molecules, heat, liquids and atoms. Physically, this movement occurs from a higher concentration region to a lower concentration region. Also, it is based on the relationship between flux and concentration gradient.

The chance in this relation determines the type of diffusion model. In the classical theory of diffusion, Fick's diffusion model is a parabolic-type partial differential equation [1-4]:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

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where κ denotes the diffusion (or thermal conductivity) coefficient. This model corresponds to the heat conduction equation constructed on the Fourier's law. Analytical solution of the diffusion (or heat) model is a Gaussian function which implies the infinite speeds of diffused particles:

$$T(x,t) = \frac{1}{(4\pi\kappa t)^{1/2}} \exp\left(-\frac{x^2}{4\kappa t}\right).$$

This is called as casuality problem and is, unfortunately an unphysical concept for many diffusion processes on different scales [5]. To overcome this weakness, generalizations of the constitutive laws for diffusion processes and the resulting partial differential equations have been studied [6].

In the generalized theory of heat diffusion, Cattaneo's law between heat flux and temperature gradient is in the following form [7,8]:

$$q + \tau_0 \frac{\partial q}{\partial t} = -k \mathrm{grad} T,$$

where τ_0 is the relaxation time. Its corresponding heat diffusion model for rigid heat conductors given by

$$\tau_0 \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = a \Delta T,$$

in which $\sqrt{a/\tau_0}$ represents the propagation speed of heat waves, is one of the generalized models frequently studied in thermal sciences. The heat waves arising from this model were called *"second sound"* [9]. The emergence of this concept has revealed that wave phenomenon is not only specific to the hyperbolic type partial differential equations. This awareness has aroused great interest among researchers, and it has led to the discovery of unnoticeable waves in parabolic-type diffusion processes [10-13].

All generalized diffusion models in classical theory are defined by integer-order derivatives based on the local description. However, it has been observed that locally generalized models are insufficient to describe the anomalous diffusion phenomenon noticed in many processes in nature. Fractional operators gain importance in eliminating this inadequacy. These operators very realistically describe the properties of memory and inheritance in many processes, thanks to their non-local definitions.

Heat conduction is one of the physical processes in which fractional operators are used most effectively. For example, the time non-local relation between heat flux and temperature gradient was first offered in the following concept by the long-tail power kernel [14-16]:

$$q(t)=\begin{cases}-\dfrac{k}{\Gamma(\alpha)}\dfrac{\partial}{\partial t}\displaystyle\int_0^t (t-\tau)^{\alpha-1}\,\mathrm{grad}T(\tau)\,d\tau, & 0<\alpha\le 1,\\ -\dfrac{k}{\Gamma(\alpha-1)}\displaystyle\int_0^t (t-\tau)^{\alpha-2}\,\mathrm{grad}T(\tau)\,d\tau, & 1<\alpha\le 2,\end{cases}$$

where Γ is the Euler's gamma function. This relation reveals the time-fractional heat conduction equation as follows:

$$\frac{\partial^{\alpha}T}{\partial t^{\alpha}}=a\Delta T,\ 0<\alpha\le 2, \tag{1}$$

where $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ denotes the Caputo fractional derivative. Eq.(1) is also known as "*diffusion-wave equation*" depending the variation of fractional order α . Similarly, the anomalous behaviors in particle jumps reveal the space or space-time fractional diffusion equations modeled with Riesz, Weyl, or fractional Laplacian operators. These models have been studied many times, both mathematically and physically [17,18].

Another non-local relation incorporating the heat flux with its history was considered by [7] as

$$q(x,t)=-\int_{-\infty}^{t} R(x,t)\nabla T(x,t-\tau)\,d\tau.$$

Assuming $R(x,t)$ as the Jeffrey's time-dependent kernel function $R(t)=\exp\left[-(t-s)/\tau\right]$, where τ is the relaxation time, yielded an integro-differential heat equation [7]:

CHAPTER 6

Optimal Homotopy Analysis of a Nonlinear Fractional-order Model for HTLV-1 Infection of CD4⁺ T-Cells

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Abstract: In this chapter, a series solution of a nonlinear fractional-order mathematical model of human T-cells lymphotropic virus-1 (HTLV-1) infection of $CD4^+$ T-cells is obtained by using a strong and capable technique so-called Homotopy Analysis Method (HAM). The proposed model is a system of nonlinear ordinary differential equations that divides $CD4^+$ T-cells into four components: uninfected cells, latently infected cells, actively infected cells and leukemia cells. The fractional model is more general than the classical one, as in the fractional model, the next state depends not only upon its current state but also upon all of its historical states. The homotopy analysis method (HAM) is applied for a strongly nonlinear fractional-order system as it utilizes a simple method to adjust and control the convergence region of the infinite series solution by using an auxiliary parameter and allows to obtain a one-parametric family of explicit series solutions. By using the homotopy series solutions, firstly, several β-curves are plotted to demonstrate the regions of convergence, then the square residual errors are obtained for different values of these regions. Secondly, the numerical solutions are presented to show the accuracy of the applied homotopy analysis method. In this chapter, a detailed proof of the convergence of this method for nonlinear fractional-order model of HTLV-1 infection of $CD4^+$ T-cells is also given. The results indicate that the HAM is accurate and capable to obtain an accurate approximate analytical solution for HTLV-1 infection of $CD4^+$ T-cells.

Keywords: Human T-cells lymphotropic virus-1 (HTLV-1) infection of $CD4^+$ T-cells, Homotopy analysis method, $\hbar$-curve, β-curves, Convergence-control parameter, Least square.

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INTRODUCTION

Human T-cells lymphotropic virus-I (HTLV-1) infection of $CD4^+$ T-cells (HLTV-1) was first discovered in 1980. The first human retrovirus, many epidemiologists, mathematicians and biologists are interested in investigating this virus due to several biological characteristics [1-2]:

1. This retrovirus shows relationship between viruses and cancer,

2. The association of HTLV-1 with a disease similar to multiple sclerosis (MS) created an opportunity to study the mechanisms that lead to the disease,

3. Its identification facilitated the discovery and isolation of the human immunodeficiency virus (HIV) [3-6], which caused a global epidemic of acquired immune deficiency syndrome (AIDS) [2].

According to the study [7], HTLV-1 is a single-stranded RNA retrovirus with reverse transcriptase activity that leads to a DNA copy of the viral genome. The viral DNA copy is then integrated into the DNA of the host genome. After integration, the viral DNA can latently persist within a T-cell for a long time. The latent infected T-cells contain the viral DNA but are not producing it, and they cannot cause new infections of susceptible cells. Stimulation of the latent infected $CD4^+$ T-cells by antigen can initiate activation of the infected cells. Actively infected T-cells can produce virus and can cause new infections of susceptible T-cells. Actively infected T-cells may then convert to adult T-cells leukemia (ATL) through certain mechanisms which are not yet known. Like HIV, HTLV-1 targets $CD4^+$ T-cells, the most abundant white cells in the immune system, decreasing the body's ability to fight infection.

In 1999, Stilianakis and Seydel [8] proposed a system of nonlinear differential equations that divides $CD4^+$ T-cells into four compartments as follows:

1. Uninfected $CD4^+$ T-cells,

2. Latently infected cells,

3. Actively infected cells,

4. Leukemia cells.

Let $T(t)$, $T_L(t)$, $T_A(t)$ and $T_M(t)$ represent the concentration of healthy CD4$^+$ T-cells at time t, latently infected cells, actively infected cells and leukemia cells, respectively. This model is formulated as follows

$$\begin{aligned}
\frac{dT}{dt} &= \Lambda - \mu_T T - kT_A T,\\
\frac{dT_L}{dt} &= kT_A T - (\mu_L + \alpha_L)T_L,\\
\frac{dT_A}{dt} &= \alpha_L T_L - (\mu_A + \rho)T_A,\\
\frac{dT_M}{dt} &= \rho T_A + \beta T_M\left(1 - \frac{T_M}{T_{M_{max}}}\right) - \mu_M T_M.
\end{aligned} \tag{1}$$

Table **1** summarizes the meaning of parameters and variables. Wang *et al*. [9] have investigated the global dynamics of system (1). Song and Li [7] investigated the dynamics behavior of the following model

$$\begin{aligned}
\frac{dT}{dt} &= \Lambda - \mu_T T - k\frac{T_A}{1+\alpha_1 T_A}T,\\
\frac{dT_L}{dt} &= k\frac{T_A}{1+\alpha_1 T_A}T - (\mu_L + \alpha_L)T_L,\\
\frac{dT_A}{dt} &= \alpha_L T_L - (\mu_A + \rho)T_A,\\
\frac{dT_M}{dt} &= \rho T_A + \beta T_M\left(1 - \frac{T_M}{T_{M_{max}}}\right) - \mu_M T_M.
\end{aligned} \tag{2}$$

Table 1. List of variables and parameters (modified from [2]).

Parameters and Variables	Meaning
Dependent variables	
T	Uninfected CD4$^+$ T-cells population concentration
T_L	Latently infected CD4$^+$ T-cells concentration
T_A	Activity infected CD4$^+$ T-cells concentration
T_M	Leukemic CD4$^+$ T-cells concentration
Parameters and constants	
μ_T	Natural death rate of CD4$^+$ T-cells concentration
μ_L	Blanket death rate of latently infected CD4$^+$ T-cells

162 *Current Developments in Mathematical Sciences*, 2022, *Vol. 3*, 162-185

CHAPTER 7

Behavior Analysis and Asymptotic Stability of the Traveling Wave Solution of the Kaup-Kupershmidt Equation for Conformable Derivative

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Abstract: This article suggests solving the traveling wave solutions of the time-fractional Kaup-Kupershmidt (KK) equation *via* $1/G'$-expansion and sub-equation methods. Non-local fractional derivatives have some advantages over local fractional derivatives. The most important of these advantages are the chain rule and the Leibniz rule. The conformable derivative, which has a local fractional derivative feature, is taken into account in this study. Different types of traveling wave solutions of the time-fractional KK equation have been produced by using the important benefits of the time-dependent conformable derivative operator. These wave types are dark, singular, rational, trigonometric and hyperbolic type solitons. 2D, 3D and contour graphics are presented by giving arbitrary values to the constants in the solutions produced by analytical methods. These presented graphs represent the shape of the standing wave at any given moment. Besides, the advantages and disadvantages of the two analytical methods are discussed and presented in the result and discussion section. In addition, wave behavior analysis for different velocity values of the dark soliton produced by the analytical method is analyzed by simulation. The conditional convergence and asymptotic stability of the dark soliton discussed are analyzed. Computer software is also used in operations such as drawing graphs, complex operations, and solving algebraic equation systems.

Keywords: $1/G'$-expansion method, Asymptotic stability, Conformable derivative, Sub-equation method, Time-fractional Kaup-Kupershmidt equation.

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Mehmet Yavuz & Necati Özdemir (Eds.)

INTRODUCTION

Recently, a number of works on nonlinear partial differential equations (NLPDEs) have been raised, as they may be employed in many disciplines, including engineering, physical, chemical and biological sciences. Most of these works have focused on attaining analytical solutions for fractional PDEs. However, fractional derivative definitions such as Caputo and Riemann-Liouville, are not always capable of reaching analytic solutions, for they do not meet some basic principles of known integer order derivatives. Some fractional derivatives are impossible to solve with these definitions.

Though the fractional derivative originated in the 17th century, interest in this subject has increased in recent years, and many studies have been made on this subject. This is because physical systems are often referred to as fractional derivatives. In the literature, several studies related to fractional derivatives, has continued to increase. Various fractional derivative definitions have been made from the 1730s to this time. Recently, Khalil et al. presented a simple, understandable and intriguing definition of the fractional derivative called the congruent fractional derivative [1].

Definition: For $t>0$ and $\alpha\in(0,1]$, an α-th order "conformable derivative" of a function is defined by (Khalil et al. 2014) as

$$T_\alpha(f)(t)=\lim_{\varepsilon\to\infty}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f\left(t\right)}{\varepsilon},$$

$$\text{for } f:[0,\infty)\to R.$$

Theorem: Let $\alpha\in(0,1]$, $t>0$ and g, f be $\alpha-$differentiable functions. Then

a) $T_\alpha(dg+cf)=dT_\alpha(g)+cT_\alpha(f)$, for all $d,c\in R$.

b) $T_\alpha(t^p)=pt^{p-\alpha}$ for all $p\in R$.

c) $T_\alpha(\lambda)=0$ for all constant functions $f(t)=\lambda$.

d) $T_\alpha(gf)=gT_\alpha(f)+f\,T_\alpha(g)$.

e) $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(g)-fT_\alpha(f)}{g^2}$.

f) If f is a differentiable function, then $T_{\alpha}(f)(t)=t^{1-\alpha}\dfrac{df}{dt}$.

Kaup has proposed first the significant diffuse classical KK equation [2] and was later presented by Kupershmidt [3]. This paper is about the investigation of the KK equation. The KK equation is used to study the operation of behavioral capillary gravitational waves and nonlinear scattered waves.

The fifth order NLPDE is as follows:

$$D_t^{\alpha}u(x,t)+ruu_{xxx}+bpu_xu_{xx}+cu^2u_x+u_{xxxxx}=0, \quad \textbf{(1)}$$

where *c, b* and *r* are real constants, $0<\alpha\le 1$ which is the parameter representing the order of fractional time derivative. We write Eq. (1) for *c=45, b=-15* and *r=-15,* in the form below [4]

$$D_t^{\alpha}u(x,t)-15uu_{xxx}-15pu_xu_{xx}+45u^2u_x+u_{xxxxx}=0. \quad \textbf{(2)}$$

Recently, great research based on the work of the classic KK equation have been done. The classical KK equation is integrable at p = 5/2 [5] and is known to have bilinear representations [4].

As a result of the effects of surface tension on phase velocity, a capillary wave is formed that travels along the phase boundary of a liquid. Besides, a longer wavelength occurring on the surface of the fluid will cause the formation of gravity-capillary waves that are affected by both the surface tension and the effect of gravity and the fluid property. As is known, the modeling of physical events is done with differential equations. Obviously, solutions of differential equations play an important role in illuminating physical phenomena. In this study, we consider the time-fractional KK equation, which is used in the modeling of capillary waves and gravity capillary waves, which have an important place physically. If the constants in the solutions we have presented gain physical meaning, it will be much more valuable.

There are many studies in the literature regarding the time-fractional KK equation. For example; the time-fractional KK equation has been solved via 2-D Legendre multiwavelet method [6], Lie point symmetries of the time-fractional KK equation are found and its invariant solutions are determined with the help of infinitesimal generators [7], with the help of extended G'/G-expansion and improved G'/G-

CHAPTER 8

Mathematical Analysis of a Rumor Spreading Model within the Frame of Fractional Derivative

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Abstract: Rumor spreading is a trivial social practice, which has a long history of affecting society both in a positive and negative way, and modelling of transmission of rumors has been an attractive area for social and, of late, for physical scientists. In this chapter, we have modified the rumor-spreading model by incorporating fractional derivatives in the Caputo sense. To analyze the spread of rumors in social as well as virtual networks, we have considered four populations, namely, ignorant, spreader, recaller, and stifler. The existence and uniqueness, and boundedness of the solutions of the present model have been exhibited theoretically. Numerically, we have experimented with the effect of fractional derivatives and the density of one population on the other population by demonstrating the impact of rumor spread with the change of various parameters.

Keywords: Adams-Bashforth-Moulton method, Caputo fractional derivative, Mathematical model, Rumor spreading.

INTRODUCTION

While there are lots of happenings that entertain us for the relaxation of our lives in this world, it is unfortunately a part of human nature to constantly prefer a chunk more. As a result, we tend to consider events that can be genuinely now no longer true. Sometimes we do it collectively for a few activities, that refer to an object, event, or subject matter of public attention and that is how rumors start. Rumors have existed as a large model of social verbal exchange and a trivial social phenomenon throughout human evolutionary history.

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People spread rumors for a variety of reasons, including raising awareness, slandering others, creating momentum, diverting attention, and inciting panic. Due to the speedy advancement in various online platforms such as Facebook, Twitter, WhatsApp, *etc.*, the spread of rumors moves from verbal to digital, and as a result, their transmission becomes faster than ever before. Rumors may thrill us, scare us or entertain us temporarily or for a long period. Many hearsay stories about some great personalities who failed in mathematics give us hope that bouncing back may happen with us also. All-time fascinating stories about alien life, confirm that we are not alone in this universe. Rumors about celebrities, politicians, and scientific, financial, and social happenings change our way of looking at events/personality. Presently, a huge racket of rumor industry is running surrounding the happenings of Covid-19 news. These misinformed and misguiding stories harm nations, and they affect the measures implemented to control the situations. Many bloggers are running their business based on rumors only. Websites like www.snopes.com, www.thecut.com, www.cisa.gov, www.factcheck.org *etc.* are some websites, which check the factuality of rumors. With the advent of the internet, rumors are now shared via instant messengers, emails, or publishing blogs. Till the late nineties, studies of the impact of rumors were the areas for only social scientists. Kimmel, in his book "Rumors and Rumor Control," has discussed the understanding and controlling of rumors [1]. But, of late, mathematicians and computer scientists have shown remarkable interest in modelling the transmission of rumors in their research works. Despite an interesting similarity between rumor spreading and the epidemic model [2, 3], rumor spreading dynamics have received less attention than epidemic spreading. Daley and Kendall were the first to investigate the issue of rumor transmission and established the DK model [4-6], where they looked into the subject of rumors by splitting the people into three groups: ignorants, spreaders, and stiflers, which corresponded respectively to those who were unaware of the rumor, those who had heard the rumor and aggressively disseminated it, and those who had heard the rumor but had stopped spreading it. The Maki-Thompson (MK) model appeared next, as one of the DK model's modifications [7]. The rumor spreading was performed via direct contacts of the spreaders with others in the MK model, which has been extensively utilized for quantitative studies of rumor spreading [8-10]. In the study [11], Zhao *et al.* considered the case of online blogging and analyzed the rumor spread with consideration of the forgetting mechanism. The author has presented a model to analyze the impact of rumors on market [12]. Based on the fulfillment of two specific criteria simultaneously for a rumor to surely invade, Galam has used a majority rule reaction-diffusion dynamics to model rumor [13]. More information on rumor spreading model can be found in [14-22] and in the references therein.

Fractional calculus, being one of the most happening areas of research in recent times, has become an exciting trend for scientists, mathematicians, and engineers to explore their areas of study in a fractional sense. This is due to the potentiality of the fractional derivatives to portray the natural and man-made complex phenomena more realistically. The invention of fractional calculus takes us way back to the year 1695, when L'Hopital asked Leibniz about the possibility that n could be something other than an integer in $(d^n f)/(dt^n)$. The search for an operator that continuously transforms f into its nth derivative or anti-derivative opens the door to a vast area of studies called fractional calculus. Many fractional derivatives such as Caputo, Grünwald Letnikov, Riemann-Liouville, Jumarie, Caputo-Fabrizio, Atangana-Baleanu are inverted thereafter and their theories are explored in a wide range [23-26]. The fascinating results obtained by analyzing the physical model by incorporating various fractional derivatives in the field of science and technology can be observed in some work presented in [27-33]. The author has analyzed Noyes–Field model for the nonlinear Belousov–Zhabotinsky reaction by incorporating fractional derivative [34]. In the studies [35, 36], authors have presented new existence results of fractional integro-differential equations in Atangana–Baleanu sense. Some very interesting works on non-singular fractional derivatives can be found in [37-39]. Mirzazadeh *et al.* [40], examined a sixth-order dispersive (3+1)-dimensional nonlinear time-fractional Schrödinger equation with cubic-quintic-septic nonlinearities. Nisar *et al.* [41], considered nonlinear Hilfer neutral fractional derivatives with a non-dense domain and analyzed controllability results. Some interesting results are derived by authors [42-54] using numerical and modified schemes. Iyiola *et al.* [55], in their work, have analyzed a generalized Chagas vectors re-infestation model of fractional order type and presented some interesting findings. Moreover, [56-62] are also interesting papers presenting illustrative applications of fractional order modeling.

Even though the rumor spreading model is an analogy to the epidemic model, a widely studied model in the literature of mathematical epidemiology, till date, no research work can be observed where these models are treated in a fractional derivative sense. Fractional-order derivative, being the generalization of the integer-order derivative, is capable of demonstrating better results in modeling real phenomena and due to this reason, in this chapter, we plan to study the burning topic like rumor spread in social as well as virtual networks in the frameworks of fractional derivatives. To model the rumor spread, the populace is classified into four classes in this chapter: ignorants, spreaders, recallers, and stiflers and analyzed the model both theoretically and numerically. Fractional derivative is considered in the Caputo sense. The impact of a fractional derivative is observed, while experimenting with the evolution of densities of various groups under the influence

CHAPTER 9

A Unified Approach for the Fractional System of Equations Arising in the Biochemical Reaction without Singular Kernel

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Abstract: The pivotal aim of the present work is to find the solution for the fractional system of equations arising in the biochemical reaction using q-homotopy analysis transform method (q-HATM). The hired scheme technique unification of Laplace transform with q-homotopy analysis method, and fractional derivative defined with Caputo-Fabrizio (CF) operator. To validate and illustrate the competence of the future method, we examined the model in terms of fractional order. The fixed-point theorem hired to demonstrates the existence and uniqueness. Moreover, the physical nature of achieved solutions has been captured in terms of plots for different order. The obtained results elucidate that the considered algorithm is easy to implement, highly methodical, and very effective as well as accurate to analyse the nature of nonlinear differential equations of fractional order arising in the connected areas of science and engineering.

Keywords: Biochemical reaction, Caputo-Fabrizio derivative, Enzyme kinetics, Mathematical model, Homotopy analysis method, Laplace transform.

INTRODUCTION

The concept of Fractional calculus (FC) was initiated in Newton's time. Nevertheless, it fascinated the attention of many authors recently. The concept of classical or integer-order is associated with power law, and it has a wide range of

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applications and admits numerous properties. However, mankind always looks for the invocation and modification for betterment to lead life systematically and happily. Recently, scientists and mathematicians notice that, the classical concept is not able to capture memory and hereditary-based consequences, and then they suggested with some essential results that the concept of FC is suitable for these types of studies. Moreover, the classical concept is a subset of FC, and we can be able to capture the behaviour of the corresponding system for classical values as a particular case. The main reason for the researchers attracted towards the study of FC is the open problems (including physical and geometrical interpretation of the concept), basic rules admitted by classical concept to extend the results and many others. Many researchers present their own notions and viewpoints with novel concepts in different forms [1-6].

Studying the biochemical models with the mathematical system is always a venue for innovation and development to understand and predict the corresponding complex behaviour of phenomena. Particularly, in biochemical systems, enzyme kinetics have been effectively exemplified with the aid of a system of ordinary differential equations [7-9]. The system was constructed uniquely on reactions that were deprived of spatial dependency of the several concentrations. Here, we consider the system studied by authors in the study [10, 11], and the corresponding enzyme reaction model is presented with enzyme E, substrate S, and product P as [12]

$$E + S \underset{\beta_{-1}}{\overset{\beta_1}{\longleftrightarrow}} ES \overset{\beta_2}{\longleftrightarrow} E + P \quad \textbf{(1)}$$

where $\beta_1,\ \beta_{-1}$ and β_2 are positive rate constants for each reaction. Here, ES the enzyme-substrate intermediate complex. Now, the reactants concentration is presented as follows for Eq. (1) $s = [S],\ c = [C], p = [P],\ e = [E]$. Then employing the law of mass action, we have [10-12] system with associated conditions

$$\frac{ds}{dt} = -\beta_1 es + \beta_{-1}c, \quad s(0) = s_0,$$
$$\frac{de}{dt} = -\beta_1 es + (k_{-1} + k_2)c, \quad e(0) = e_0, \quad \textbf{(2)}$$

$$\frac{dc}{dt} = k_1 es - (k_{-1} + k_2)c, \ \ c(0) = c_0,$$

$$\frac{dp}{dt} = k_2 c, \quad p(0) = p_0.$$

By the assist of Eq. (2), authors in [12] derived the following system

$$\begin{aligned} \frac{d\mathcal{P}(t)}{dt} &= -\alpha\mathcal{P} + \alpha(\mathcal{P} + \beta - \varepsilon)\mathcal{Q}, \\ \frac{d\mathcal{Q}(t)}{dt} &= \mathcal{P} - (\mathcal{P} + \beta)\mathcal{Q}, \\ \frac{d\mathcal{S}(t)}{dt} &= \varepsilon\mathcal{Q}. \end{aligned} \tag{3}$$

The projected system is analyzed by many researchers to illustrate their viewpoints using numerical schemes. For instance, variational iteration scheme [10], multistage homotopy analysis algorithm, Adomian decomposition [13] and multistage homotopy-perturbation techniques [14] and others.

The fractional-order derivatives are familiarized by Leibnitz soon after the classical concept. As compared to classical calculus, it was soon discovered that fractional calculus (FC) is more suitable capturing complex phenomena [15-25]. The FC considered is the essential apparatus to illustrate the chemical and biological phenomena. Most of the mathematical models demonstrate the non-local distributed effects, hereditary properties and system memory. These properties are necessary to describe the above-cited phenomena. The pivotal aim of generalizing the integer to fractional order is to capture consequences related to non-locality, long-range memory and time-based properties and also anomalous diffusion aspects [26-31]. Most familiarly hired operators to analyze many models are Riemann, Liouville, Caputo, Fabrizio and others [1-6, 32, 33]. In this connection, Caputo and Fabrizio in 2015 overcome the many limitations raised by many mathematicians to generalize complex models, and then many scholars hired to present simulating consequences. It has been proved by many researchers that, the CF fractional operator has great results compared to other fractional operators.

In the present work, we consider the fractional-order system in order to include all the above-described consequences into the system cited in Eq. (3) and which as follows

CHAPTER 10

Floating Object Induced Hydro-morphological Effects in Approach Channel

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Abstract: Transversal and diverging waves, return flows, propeller induced jet flows, and other hydrodynamic effects induced by a floating object may cause significant movement and/or suspension of bottom and bank sediments in the marine environment, especially in approach channels. Using the CFD (Computational Fluid Dynamics) process, the hydro-morphodynamic effects induced by a non-powered floating object navigating in an approach channel are investigated in this study. The approach channel dimensions depth, width, and channel slope are determined according to PIANC (2014) [1]. The floating object locations and velocities are used in nine different scenarios. In these cases, the floating object is 0.90, 1.10, and 1.30 meters from the bottom of the approach channel, respectively. According to the findings, when the floating object is located nearest to the bottom and its speed is fastest, there is a significant amount of sediment suspension and sediment movement in the channel slope, which is mostly attributed to super-critical return flows. When the floating object is farthest from the channel bottom and the floating object speed is lowest, however, there is a noticeable reduction in the acceleration and suspension of the sediment. As a result, the velocity and location of the floating object, channel slope, the kinematics of ship-generated waves, and particularly the return flows are found to have a significant impact on sediment movement and suspension.

Keywords: CFD, Floating object, Hydrodynamic, Morphodynamic, Sediment suspension, Sediment transport.

INTRODUCTION

Waterborne commerce has increased continuously over the last decades and this situation has led to an increase in ship dimensions and ship numbers. Because of

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the increasing ship sizes and the number of ships, safe navigation and economical requirements have gained great importance. In order to ensure safe navigation in the approach channel, the channel must be wide and deep enough for vessel traffic, but it is not so deep or large as to require excessive dredging. Therefore, vessel or floating object-induced sediment transport must be well studied in the design phase of the approach channel.

Floating objects navigating in a natural or an artificial approach channel, in rivers or inland waterways cause several hydrodynamic disturbances in the form of waves and currents. Floating objects generate two main types of waves, namely primary (drawdown) and secondary waves [2]. The primary wave system consists of significant water level depression along the hull of the floating object and return flow. The primary wave system is dominant where floating object induced the cross-sectional blockage is significant [3]. Secondary waves are gravity waves generated by pressure peaks along the floating objects and these waves are short waves. The secondary waves are dominant in canals for ocean-going floating object and on most rivers, where the blockage factors of the floating object are usually very low [4].

In previous, many researchers have worked on ship-generated waves [5-7]. These studies show that there is a relationship between wave height and ship type, draught, speed, and distance to the banks. Ship-generated waves cause intensive sediment resuspension and sediment transport in the approach channel. These actions are so important that Houser (2011) found that the vessel-generated wakes (including drawdown and surge waves) have much more effects on sediment resuspension than wind waves and suspended sediment concentration (SSC) increases with the increment of turbulent kinetic energy (TKE) of the supercritical pilot-boat wakes [8].

Floating object-induced hydrodynamic effects such as water level drawdown, transversal, and diverging waves, return flows, propeller jet flows, etc., lead to bank erosion, sediment resuspension, and environmental impact on plankton, fish, plants, etc. Rapaglia *et al.*, (2011) measured water velocity, water depth, and sediment concentration on the shoals alongside the shipping channel after the passage of forty vessels. They found that higher return velocities and ten vessel-induced wakes led to SSC concentration above 400 mg/L, which is 30 times higher than the average background concentration [9]. Ji *et al.*, (2014) investigated that ship induced suspended particulate matter (SPM) concentration navigating in an approach channel with and without a propeller [10]. Schroevers *et al.* (2015) carried out a 1/1 scale physical experiment using a heavy loaded barge to observe the canal bottom stability in the 36 km long Juliana Canal in the south of the Netherlands. During each passage of the barge, the flow velocities under the ship and the bed change were measured. They found that the amount of erosion in the middle of the channel reached 1 cm at each 10 passages of the barge and 6 cm erosion value in total at the end of the experiment (60 passages) [11]. McConchie and Toleman (2003) investigated boat wakes-induced riverbank erosion. They measured wake wave characteristics and suspended sediment concentration at several sites, and they found that boat wakes 2-80 times larger than background wind-generated waves. So, boat-generated waves are more erosive than wind-generated waves in riverine environments, particularly where fetch lengths are restricted [12].

Also, several studies have been carried out on various parameters such as water depth, turbulence energy, ship type, ship velocity, eddies, etc., that may affect sediment resuspension. Smaoui *et al.*, (2011) investigated the quantitatively and relatively accurate relationship between sediment transport and boat traffic via a one-dimensional vertical model [13].

Some researchers have studied ship-induced current with the help of physical experiments [14 - 17]. Maynord (2000) investigated the physical forces under the ship to determine ship-induced sediment transport and sediment suspension. Lenselink (2011) studied loaded barges and investigated the velocity profile under the barge and its effect on the seabed [14]. In this study, the hydro-morphodynamic effects caused by a non-powered floating object navigating in an approach channel are investigated using a 3-D numerical model.

MATERIALS AND METHODS

Numerical Model

FLOW-3D software was used in this analysis to build a 3D hydro-morphodynamic numerical model. Flow Science, Inc. created FLOW-3D, a commercial software kit. Flow-3D uses a finite volume approach to solve the continuity equation (Eq. (1)) and the unsteady Reynolds-averaged Navier-Stokes equations governing fluid motion (Eq. (2)) [18].

$$\frac{\partial}{\partial X_i} U_i A_i = 0, \quad \textbf{(1)}$$

$$\frac{\partial U_i}{\partial t} + \frac{1}{V_f}\left(U_j A_j \frac{\partial u}{\partial X_j}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial X_i} + G_i + f_i, \quad \textbf{(2)}$$

SUBJECT INDEX

D

E

F

G

H

I

K

L

M

N

O

P

R

S

T

V

W

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