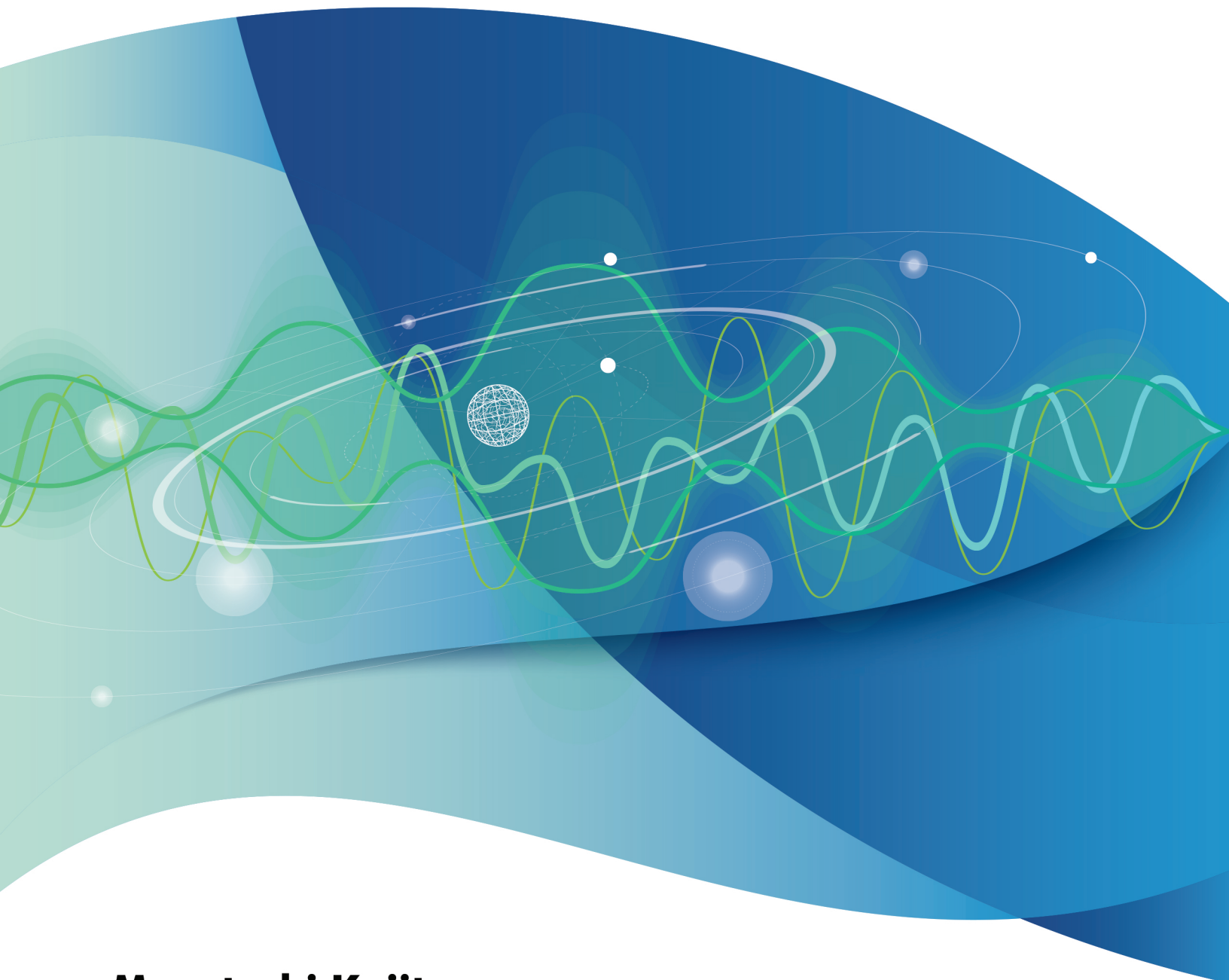


# FUNDAMENTALS OF ANALYSIS IN PHYSICS



**Masatoshi Kajita**

**Bentham Books**

# **Fundamentals of Analysis in Physics**

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## **Fundamentals of Analysis in Physics**

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ISBN (Online): 978-981-5049-10-7

ISBN (Print): 978-981-5049-11-4

ISBN (Paperback): 978-981-5049-12-1

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## **PREFACE**

This book summarizes the analysis in whole fields of physics without using the special functions, targeting college/bachelor students. Many beginners feel that it is difficult to learn each field of physics (classical mechanics, electromagnetism, quantum mechanics, relativistic quantum mechanics, statistic mechanics) in detail separately. It would be preferable to learn the whole fields as quick as possible and have a simple imagination about the relation between different fields. After learning the position of each field in the physics, it becomes easier to learn detailed parts of each field. In this book, the fundamental of mathematical treatments are introduced, which are important for the analysis in physics but not familiar to all the readers. The fundamental of the analysis in each field of physics are summarized afterwards. The estimation of measurement uncertainty is also introduced. The important points of whole physics are summarized within 150 pages.

It would be my great pleasure, if this book can help college students to understand the fundamental of physics. And I believe, it is also useful for researchers to develop new research fields.

### **CONSENT FOR PUBLICATION**

Not applicable.

### **CONFLICT OF INTEREST**

The author confirms that there is no conflict of interest.

### **ACKNOWLEDGEMENTS**

The research activity of the author is supported by a Grant-in-Aid for Scientific Research (B) (Grant No. JP 17H02881 and JP20H01920) and a Grant-in-Aid for Scientific Research (C) (Grant Nos. JP 17K06483 and 16K05500) from the Japan Society for the Promotion of Science (JSPS). I want to thank Editage ([www.editage.com](http://www.editage.com)) for English language editing.

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**CHAPTER 1****Fundamentals of Mathematical Treatments**

**Abstract:** Physical phenomena can be understood by solving equations that lead to physical laws. The first objective of this chapter is to solve certain equations that are required for physical analysis. First, an iterative solution of the equation is introduced. Using this approach, the numerical solution of an equation  $f(x) = 0$  can be obtained also when the function  $f(x)$  is too complicated for the solution to be obtained as an explicit formula.

Many physical equations can be expressed using differential and integral mathematical representations, which might not be familiar to all college students. The fundamental concepts of the differential and integral were introduced. Several fundamental mathematical formulae are reviewed.

The second objective is to solve the differential equations that are required for the physical analysis. First, some solutions of simple differential equations given by explicit formulas are introduced, which are important for their physical interpretation. However, the equations for technical use are generally too complicated. Several methods for obtaining numerical solutions are introduced, which are useful for analyzing motion orbits. There is also a phenomenon that cannot be predicted by solving equations, which is called “chaos.”

Finally, the fundamentals of the eigenvalues of matrices are introduced, which are important for understanding quantum mechanics. This chapter was prepared for undergraduate students who are not familiar with differential and integral calculus and matrices.

**Keywords:** Iterative solution, Differential, Partial derivative, Integral, Taylor expansion, Euler method, Middle point method, Runge-Kutta method, Chaos, Lyapunov exponent, matrix, Determinant, Eigen value, Eigen vector.

**1.1. INTRODUCTION**

Many physical laws have been established based on equations, and physical phenomena are predicted by solving them. It is not always a simple task to solve these equations. For example, the Newtonian law of universal gravitation is given

by the simple formula. However, it is not easy to analyze the motion of astronomical bodies when considering the interaction between more than three bodies.

The technical development of any analysis plays an important role in physics. For some cases, the simplification of equations makes it possible to obtain a solution that is expressed as simple formulas, which can provide new physical insights. However, numerical calculations make it possible to obtain the solutions of equations which cannot be expressed as a rigorous formula. The technical development of numerical calculations is important to minimize the error of solutions and the calculation time.

Physicists and engineers have many techniques for obtaining reliable solutions to equations. Different kinds of approximations have been developed in the field of theoretical physics, which are not acceptable for mathematicians. This chapter summarizes the fundamentals of solving the equations analytically and numerically.

Reference [1] seems to be useful to learn the fundamental of mathematics more in detailed for students, who are interested with physical analysis. Reference [2] seems to be readable also for high school students [2].

## 1.2. ITERATIVE SOLUTION OF EQUATIONS

There are many cases for which the solution  $x_0$  that satisfies the following equation must be obtained:

$$f(x_0) = 0 \quad (1.2.1)$$

Considering the following simple equations, the solutions are given by:

$$f(x) = ax + b \rightarrow x_0 = -\frac{b}{a} \quad (1.2.2)$$

$$f(x) = ax^2 + bx + c \rightarrow x_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (1.2.3)$$

In many other cases, the solutions cannot be described using simple formulas. Therefore, solutions are often obtained using an iterative method. When  $f(x_a) > 0$   $f(x_b) < 0$  with  $x_a < x_b$ , the solution  $x_0$  is  $x_a < x_0 < x_b$ . Then we calculate  $f(x_c)$  with  $x_a < x_c < x_b$ . If  $f(x_c) > 0$ ,  $x_c < x_0 < x_b$  is obtained. By repeating

this calculation, the possible region of  $x_0$  becomes narrower. It seems useful to choose  $x_c$  as follows:

$$x_c = \frac{|f(x_b)|x_a + |f(x_a)|x_b}{|f(x_b)| + |f(x_a)|} \quad (1.2.4)$$

because  $x_c$  is expected to be close to  $x_0$ , assuming that  $f(x)$  is approximately linear in a limited region of  $x$ . For example, we obtain the solution of  $f(x) = \cos(x) - x = 0$  using the following procedure.

$$\begin{aligned} f(0) &= 1 > 0, f(1) = -0.46 < 0, & 0 < x_0 < 1 \\ f(0.685) &= 0.089 > 0, f(1) = -0.46 < 0, & 0.685 < x_0 < 1 \\ f(0.736) &= 0.005 > 0, f(0.75) = -0.018 < 0, & 0.736 < x_0 < 0.75 \\ f(0.738) &= 0.00018 > 0, f(0.74) = -0.0015 < 0, & 0.738 < x_0 < \\ &0.74 \\ f(0.739) &= 0.00014 > 0, f(0.7395) = -0.00069 < 0, & 0.739 < \\ &x_0 < 0.7395 \end{aligned}$$

The value of  $x_0$  was obtained with an uncertainty below 0.03 %, and this uncertainty was further reduced by continuing the calculation with a narrower region of  $x$ . Biological evolution is the search for a solution for survival for a change of circumstances, which is similar to the iterative method. Evolution is possible in many different directions. Species with the appropriate adaptations (advantageous traits for survival in a new circumstance) can survive, but those with disadvantageous traits will not. Therefore, biological species with desirable adaptations dominate for a long period after the change in circumstance.

The iterative method cannot be used for discontinuous functions (*e.g.*,  $1/x$  at  $x = 0$ ). Note also that the iterative method cannot be used for cases without solutions. There may also be multiple solutions, and the iterative method should be performed with a limited region of  $x$ .

### 1.3. DIFFERENTIAL AND INTEGRAL EQUATIONS

Before introducing differential equations, we summarize the characteristics of the differential and integral. Considering a function  $y(x)$ , the differential of  $y$  is defined by (see Fig. 1.1):

**CHAPTER 2****Analysis in the Classical Mechanics**

**Abstract:** Equations of motion with arbitrary potential fields are described by simple formulas based on Newtonian law. However, they are generally complicated to solve. This chapter presents several methods for deriving and solving equations of motion with different coordinate systems.

Two-body motion is introduced based on the coordinates of the motion of the center of mass and the relative motion. Relative motion perpendicular to a relative position is discussed using angular momentum, wherein the temporal change is given by the torque. The Lagrange equation is introduced for application to all types of coordinates. The polar coordinate system is convenient for deriving an equation of motion with a spherically symmetric potential because the angular momentum is constant. The motion in the radial direction should be considered taking the centrifugal force potential into account. The Lagrangian is also introduced with respect to electromagnetic fields.

**Keywords:** Angular momentum, Center of mass, Centrifugal force, Energy, Gravity, Kepler's law, Lagrange equation, Momentum. Inertial moment, Newtonian mechanics, Relative motion.

**2.1. FUNDAMENTAL OF CLASSICAL MECHANICS**

Newton derived three laws of motion in his monumental work *Philosophiæ Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy), which was published in 1687 [1].

1. An object either remains at rest or continues to move in a straight line at a constant velocity unless acted on by a net external force.
2. The sum of the forces  $\vec{F}$  acting on an object is equal to the mass  $m$  of that object multiplied by the acceleration  $\vec{a}$  of the object,
3. When a body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

Using the Cartesian coordinate  $\vec{r} = (x, y, z)$ , the second law is given by the equation

$$m \frac{d\vec{v}}{dt} = m \frac{d^2\vec{r}}{dt^2} = \vec{F} \quad \vec{v} = \frac{d\vec{r}}{dt} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \quad \vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} \quad (2.1.1)$$

where  $m$  is the mass and  $t$  is the time. The first law corresponds to the second law with  $\vec{F} = 0$ , although it was initially given as the definition of the “inertial frame”. In classical mechanics, momentum is defined as follows (this definition is not correct in the case of relativistic theory and quantum mechanics):

$$\vec{p} = m\vec{v} \quad (2.1.2)$$

From the force, the potential energy  $P_E$  is defined as:

$$P_E = - \int F_x dx - \int F_y dy - \int F_z dz \quad (2.1.3)$$

Then eq. (2.1.1) can be rewritten as:

$$\frac{d\vec{p}}{dt} = -\nabla P_E \quad \nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad (2.1.4)$$

The temporal change of  $|\vec{p}|^2$  by a force is given by:

$$\begin{aligned} \frac{d|\vec{p}|^2}{dt} &= 2p_x \frac{dp_x}{dt} + 2p_y \frac{dp_y}{dt} + 2p_z \frac{dp_z}{dt} \\ \text{using } \frac{dp_q}{dt} &= -\frac{\partial P_E}{\partial q}, p_q = m \frac{dq}{dt} \\ &= -2m \frac{\partial P_E}{\partial x} \frac{dx}{dt} - 2m \frac{\partial P_E}{\partial y} \frac{dy}{dt} - 2m \frac{\partial P_E}{\partial z} \frac{dz}{dt} \end{aligned} \quad (2.1.5)$$

Defining the kinetic energy as follows:

$$K_E = \frac{|\vec{p}|^2}{2m} = \frac{m|\vec{v}|^2}{2} \quad (2.1.6)$$

the temporal change of the potential energy and the total energy  $E = K_E + P_E$  are given by:

$$\frac{dP_E}{dt} = \frac{\partial P_E}{\partial t} + \left[ \frac{\partial P_E}{\partial x} \frac{dx}{dt} + \frac{\partial P_E}{\partial y} \frac{dy}{dt} + \frac{\partial P_E}{\partial z} \frac{dz}{dt} \right] = \frac{\partial P_E}{\partial t} - \frac{dK_E}{dt} \quad (\text{see eq. (1.3.10)})$$

$$\frac{dE}{dt} = \frac{dK_E}{dt} + \frac{dP_E}{dt} = \frac{\partial P_E}{\partial t} \quad (2.1.7)$$

The total energy is conserved when the  $P_E$  has no dependence on time (constant potential energy distribution). Considering the interaction between two bodies  $a$  and  $b$ , the sum of their momenta is conserved because the change in momentum is given by:

$$\begin{aligned} \Delta \vec{p}_a &= \int \vec{F}_{b \rightarrow a} dt & \Delta \vec{p}_b &= \int \vec{F}_{a \rightarrow b} dt \\ \vec{F}_{a \rightarrow b} &= -\vec{F}_{b \rightarrow a} \quad (\text{from the Newtonian third law}) \end{aligned} \quad (2.1.8)$$

According to the theory of relativity and quantum mechanics, the correspondence between the three-dimensional position vector  $\vec{r}$  and the momentum vector  $\vec{p}$  can be expanded to the relation between the four-dimensional position vectors  $\vec{r}^4 = (x, y, z, ct)$  and  $\vec{p}^4 = (p_x, p_y, p_z, E/c)$ , where  $c$  is the speed of light in a vacuum. For two interacting bodies, the sum of the four-dimensional momentum vector must be conserved. What is the correspondence between time and energy? Equation (2.1.4) shows that the change in  $\vec{p}$  is given by the dependence of  $P_E$  on  $\vec{r}$ . However,  $E$  changes when there is a temporal change in  $P_E$ . We can say that the components of  $\vec{p}^4$  change when the potential energy changes due to the corresponding components of  $\vec{r}^4$ .

## 2.2. TWO-BODY MOTION AND TORQUE

Here, we consider two-body motion (masses:  $m_1$  and  $m_2$ ) based on the motion of the center of mass, binding, and rotation. Bodies 1 and 2 experience the external forces  $\vec{F}_1 (= \delta \vec{F}_1 + \vec{F}_{int})$  and  $\vec{F}_2 (= \delta \vec{F}_2 - \vec{F}_{int})$ , respectively, as shown in Fig. (2.1). Here,  $\pm \vec{F}_{int}$  is the interaction force between the two bodies.

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = \vec{F}_1 \quad (2.2.1)$$

## CHAPTER 3

# Fundamental Meaning and Typical Solutions of Maxwell's Equations

**Abstract:** The main objective is to understand the meaning of Maxwell's equations. It is a set of four differential equations that describe several fundamental laws that are already known. Why did these equations cause a revolution in physics? One reason is that the distribution of the electromagnetic field can be obtained by solving first- or second-order differential equations. Several examples have been shown to obtain the distribution of the electric field produced by simple electrodes. The three-dimensional trapping of charged matter can be achieved via two methods: the combination of a DC electric field and a DC magnetic field or using an AC electric field.

The most important aspect of Maxwell's equation is that it elucidates the identity of light as an electromagnetic wave. Light energy is given as the potential energy of electric and magnetic fields. Light has momentum and generates radiation pressure on a reflecting mirror.

The speed of light was determined to be independent of the observer, which was the basis for the theory of relativity. A fundamental aspect of the theory of special relativity is also introduced.

**Keywords:** Ampere's law, Coulomb's law, Cyclotron radiation, Electromagnetic wave, Faraday's law of induction, Gauss's theorem, Maxwell's equation, Lorenz transform, Stokes's theorem, The speed of light, Theory of relativity, Trap of charged matter.

### 3.1. WHAT ARE MAXWELL'S EQUATIONS?

The fundamentals of electromagnetism regarding an electric field  $\vec{E}$  and magnetic field  $\vec{B}$  are summarized by Maxwell's equations as follows (in SI units) [1]:

$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon}$  ( $\rho$ : the electric charge density,  $\epsilon$ : the permeability)

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon} \quad (3.1.1)$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t}, \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t}, \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad (3.1.2)$$

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$$\nabla \cdot \vec{B} = 0$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (3.1.3)$$

$$\nabla \times \vec{B} = \mu \left[ \vec{j} + \varepsilon \frac{\partial \vec{B}}{\partial t} \right] \quad (\vec{j}: \text{the electric current density, } \mu: \text{the permittivity})$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu \left[ j_x + \varepsilon \frac{\partial E_x}{\partial t} \right], \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} = \mu \left[ j_y + \varepsilon \frac{\partial E_y}{\partial t} \right], \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu \left[ j_z + \varepsilon \frac{\partial E_z}{\partial t} \right] \quad (3.1.4)$$

As shown in the following, these equations are novel expressions of laws that were already known.

Equation (3.1.1) is derived from Coulomb's law [2] for the electric field given by a point electric charge  $q_e$

$$\vec{E} = \frac{q_e}{4\pi\varepsilon|\vec{r}|^3} \vec{r} \quad (3.1.5)$$

The integral of  $\vec{E}$  over the surface surrounding the electric charge is given as follows using polar coordinate ( $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \theta$ ):

$$\iint \vec{E} \cdot d\vec{S} = \iint |\vec{E}| r^2 \sin \theta d\theta d\phi = \frac{q_e}{\varepsilon} \quad (3.1.6)$$

From Gauss's law [3], we have:

$$|\vec{E}| = \int \frac{\partial}{\partial r} |\vec{E}| dr$$

$$\iint \vec{E} \cdot d\vec{S} = \iiint \left[ \frac{\partial |\vec{E}|}{\partial r} \right] dV = \frac{q_e}{\varepsilon} \quad dV = r^2 \sin \theta dr d\theta d\phi$$

$$\text{using } \frac{\partial}{\partial r} = \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} + \frac{\partial z}{\partial r} \frac{\partial}{\partial z} = \sin \theta \cos \phi \frac{\partial}{\partial x} + \sin \theta \sin \phi \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial z}$$

$$E_x = \sin \theta \cos \phi |\vec{E}|, E_y = \sin \theta \sin \phi |\vec{E}|, E_z = \cos \theta |\vec{E}|$$

$$\frac{\partial |\vec{E}|}{\partial r} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \nabla \cdot \vec{E}$$

$$\iint \vec{E} \cdot d\vec{S} = \iiint \nabla \cdot \vec{E} dV = \frac{q_e}{\varepsilon} = \iiint \frac{\rho}{\varepsilon} dV \quad (3.1.7)$$



and eq. (3.1.1) is derived. Since there is no magnetic charge, eq. (3.1.3) is derived from eq. (3.1.1) by changing  $\vec{E} \rightarrow \vec{B}, \rho \rightarrow 0$ .

Before explaining eqs. (3.1.2) and (3.1.4), Stokes's theorem [4] for the line integral of a vector quantity should be introduced. The line integral is a vector quantity, for which the  $x$ ,  $y$ , and  $z$ -components result from the analyses of the  $yz$ ,  $zx$ , and  $xy$ -planes, respectively. Here, we consider the component in the  $z$ -direction (line integral in the  $xy$ -plane). The line integral of the closed loop of a vector quantity  $\vec{f}$  is given by

$$\oint \vec{f} \cdot d\vec{l}_z = \oint f_x dx + \oint f_y dy \quad (3.1.8)$$

Stokes's theorem shows that:

$$\oint \vec{f} \cdot d\vec{l}_z = \iint_{\text{inside loop}} \left[ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right] dS_z \quad dS_z = dx dy \quad (3.1.9)$$

The detailed derivation of eq. (3.1.9) is represented in Ref. [4]. As a simple case, we consider the line integral of a loop  $(x_0, y_0) \rightarrow (x_1, y_0) \rightarrow (x_1, y_1) \rightarrow (x_0, y_1) \rightarrow (x_0, y_0)$ . Then, we have:

$$\begin{aligned} \oint \vec{f} \cdot d\vec{l}_z &= \int_{x_0}^{x_1} f_x(x, y_0) dx + \int_{y_0}^{y_1} f_y(x_1, y) dy + \int_{x_1}^{x_0} f_x(x, y_1) dx + \int_{y_1}^{y_0} f_y(x_0, y) dy \\ &= \int_{x_0}^{x_1} [f_x(x, y_0) - f_x(x, y_1)] dx + \int_{y_0}^{y_1} [f_y(x_1, y) - f_y(x_0, y)] dy \\ &= \int_{x_0}^{x_1} \int_{y_0}^{y_1} \left[ \frac{\partial f_y(x, y)}{\partial x} - \frac{\partial f_x(x, y)}{\partial y} \right] dx dy \end{aligned} \quad (3.1.10)$$

Next, we consider the vector quantity in the tangential direction of the circular loop,  $\vec{f} = (f_x, f_y) = f(r)(-\sin \theta, \cos \theta)$ , using the two-dimensional polar coordinate ( $x = r \cos \theta, y = r \sin \theta$ ). Then using:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial \tan \theta}{\partial x} \frac{\partial \theta}{\partial \tan \theta} \frac{\partial}{\partial \theta} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \quad (3.1.11)$$

$$\iint \left[ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \right] d\vec{S}_z = \int_0^{2\pi} d\theta \int_0^r \left[ \frac{\partial f(r)}{\partial r} + \frac{f(r)}{r} \right] r dr = 2\pi r f(r) = \oint \vec{f} \cdot d\vec{l}_z \quad (3.1.12)$$

is derived (see Fig. 3.1).

## Fundamentals of Analysis in Quantum Mechanics

**Abstract:** Light has the dual characteristics of particles (photons) and electromagnetic waves. The photon has an energy of  $E = h\nu$  ( $\nu$ : frequency,  $h$ : Planck constant) and the momentum of  $\vec{p} = h\vec{k}$  ( $\vec{k}$ : wavenumber and  $\frac{1}{|\vec{k}|}$  is the wavelength). The photon density is proportional to the square of the amplitude of the electromagnetic waves. The fundamental aspect of quantum mechanics is that these characteristics apply to all matters. The properties of matters are described by wave functions. The probability of the existence of the matter is proportional to the square of the associated wave function. When a matter is localized in a limited region, it can only assume discrete values of energy because the wavelength of the matter wave must be an integral division of the region. The phase of the wavefunction has uncertainty on order  $1/2$  radians; therefore, position and momentum (time and energy) cannot be simultaneously determined. As the size of the localization area of the wavefunction becomes smaller, the minimum kinetic energy becomes larger because of the smaller wavelength (larger momentum uncertainty).

The Schroedinger equation was derived based on the idea that the relationship between the frequency and the wavenumber corresponds to that between energy and momentum given by classical mechanics, which makes it possible to obtain the wave functions of matters in the energy eigenstates. Several examples of solutions to the Schroedinger equation are introduced. The mixture between different energy eigenstates and the shift in the energy eigenvalues are induced by electromagnetic fields. The temporal change of the wave function (transition between different energy states) is also obtained using the Schroedinger equation.

**Keywords:** Adiabatic rapid passage, Backbody radiation, Bohr radius, Boson, Eigenfunction, Eigenvalue, Electric induced transparency (EIT), Fermion, Operator, Particle-wave duality, Photoelectronic effect, Rabi oscillation, Schroedinger equation, Stark shift, Uncertainty principle, Zeeman shift.

### 4.1. ESTABLISHMENT OF QUANTUM MECHANICS

As shown in chapter 3.4, the characteristics of light were determined to be identical to that of an electromagnetic wave. However, some phenomena could not be explained by considering light only as a wave. The spectrum distribution of the blackbody radiation (radiation from objects having finite temperature) has the following characteristics [1].

- (1) The energy density at the low frequency region is given by  $\frac{8\pi\nu^2}{c^3}k_B T$ , where  $\nu$  is the frequency,  $T$  is the thermodynamic temperature  $T$  (see chapter 6.2), and  $k_B$  is the Boltzmann constant (defined to be 1.380649 J/K at 2019 [2]). This characteristic is explained by the pure wave characteristic of the radiation.
- (2) With the high frequency region, the distribution is proportional to  $\nu^3 \exp\left[-\frac{h\nu}{k_B T}\right]$ . Here,  $h$  is the Planck constant (defined to be  $6.62607004 \times 10^{-34}$  J/Hz at 2019 [2]). This characteristic cannot be explained considering light only as a wave.

Planck solved this mystery based on the assumption that light energy can only assume  $n_\alpha h\nu$  with the probability proportional to  $\exp\left(-\frac{n_\alpha h\nu}{k_B T}\right)$  (see chapter 6.3), where  $n_\alpha (\geq 0)$  is integer. The energy density is given by:

$$P_{BBR} = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp\left[\frac{h\nu}{k_B T}\right] - 1}$$

$$h\nu \ll k_B T \quad P_{BBR} \approx \frac{8\pi\nu^2}{c^3} k_B T \quad (\text{influence of the energy gap is small})$$

$$h\nu \gg k_B T \quad P_{BBR} \approx \frac{8\pi h\nu^3}{c^3} \exp\left[-\frac{h\nu}{k_B T}\right] \quad (4.1.1)$$

Which is in good agreement with the experimental result. However, the validity of Planck's assumption was not confirmed at that time.

In addition, the emission of electrons from a material was observed when irradiated with incident light (called the photoelectric effect). The experimental results for the photoelectric effect show that the energy of the emitted electrons is independent of the intensity of the light, although the number of emitted electrons is proportional to the intensity. Moreover, emission does not occur when  $\nu$  is lower than a minimum threshold value  $\nu_{min}$  and the energy of the emitted electron is proportional to the  $\nu - \nu_{min}$  [3]. Einstein proposed a new concept of wave-particle duality; light has the characteristics of both waves and particles. The energy of each particle (called a photon) is  $E = h\nu$  and the momentum in the propagation direction is  $|\vec{p}| = h|\vec{k}| = \frac{h}{\lambda}$  (as shown in chapter 3.4), where  $\vec{k}$  is the wavenumber vector and  $\lambda$  is the wavelength. Planck's assumption was also explained by the particle-wave duality

of light. This duality was a special characteristic of light until the concept of matter waves was proposed.

There has also been a mystery regarding atomic structure since the discovery of electrons [4]. Atoms must have a structure that includes electrons, which cannot be explained by classical mechanics. Rutherford's scattering experiment [5] revealed that atoms consist of a nucleus with a positive charge and negatively charged electrons that orbit the nucleus. However, the circular motion of electrons leads to the emission of radiation energy and the corresponding loss of kinetic energy (see chapter 3.5). Electrons should crash into the nucleus after losing kinetic energy. Therefore, it was a mystery that electrons remain in their orbit. There was also another mystery in that only discrete frequency components were observed from the emission of hydrogen atoms [6].

Bohr established the "Old Quantum Mechanics" in 1913, assuming that particles bounded in a limited region of  $q$  must satisfy the following relationship [7].

$$\oint p_q dq = nh \quad n: \text{integer}, p_q: \text{momentum in the } q\text{-direction} \quad (4.1.2)$$

Applying this assumption to the electron in orbit (radius  $r$  and velocity  $v$ )

$$2\pi r p = 2\pi r \mu_e v = nh \quad (4.1.3)$$

where  $\mu_e$  is the reduced mass between the electron and the nuclear. From the balance between the centrifugal force and the Coulomb force,

$$\frac{\mu_e v^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} \quad e: \text{unit electric charge} \quad (4.1.4)$$

From eqs. (4.1.3-4), the possible radius of the electron orbit is given by:

$$r = a_B n^2 \quad a_B = \frac{\epsilon_0 h^2}{\pi \mu_e e^2} \quad (4.1.5)$$

where  $a_B$  is called the Bohr radius. The possible electron energy is given by:

$$E_e = \frac{\mu_e}{2} v^2 - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{e^2}{8\pi\epsilon_0 a_B} \frac{1}{n^2} \quad (4.1.6)$$

**CHAPTER 5****Relativistic Quantum Mechanics and Spin**

**Abstract:** Spin is one of the most important characteristics of particles, which results in a fine and hyperfine energy structure and Zeeman energy shift. Spin is a property of particles, independent of the density distribution given by the potential field. The spin eigenfunction should be described using a vector, and the spin operator is given by a matrix. The property of spin is not derived from the Schroedinger equation, which treats wave functions as scalars.

The Dirac equation was developed to obtain wave functions from the relationship between frequency (energy) and wavenumber (momentum) given by the theory of special relativity, for which the equation is given by the  $4 \times 4$  matrix and the wave functions are given by four-dimensional vectors. There are four solutions for one equation, which correspond to  $\pm \frac{1}{2}$  spin states and the positive and negative rest energies. The existence of negative rest energy was confirmed by the discovery of positrons (antiparticles). The Zeeman shift induced by spin is derived from the exchange of matrix production.

**Keywords:** Antiparticle, Dirac equation, Electron sea, Electron spin, Fine structure, Klein-Gordon equation, Lamb shift, Pauli matrix, Quantum electrodynamics, Relativistic effect, Zeeman shift.

**5.1. ELECTRON SPIN**

As shown in chapter 4, the quantum energy state of an electron in an atom is derived from the Schroedinger equation using the principal quantum number  $n$ , rotational quantum number  $L$ , and magnetic quantum number  $M$ . These quantum numbers are given by the motion of electrons under a Coulomb potential owing to the nucleus. An electron is a Fermion, and only one electron can be in a quantum state (chapter 6.6). However, two electrons can be in an  $(n, L, M)$  state, which means there are two states with electrons. In the Stern-Gerlach experiment, an Ag atomic beam traversed through an area with an inhomogeneous magnetic field. The atoms were deflected from the straight path in two opposite directions. The deflection angle was the only quantized parameter [1]. The electron orbital angular momentum of the Ag atom was zero. This result shows that the electron has two states like a permanent magnet: the S-pole or N-pole in the direction of the magnetic field (see chapters 4.4 and 4.5). In analogy with the  $M$  states defined as  $-L \leq M \leq L$  for each  $L$  state (number of states  $2L + 1$ ), these two states of electrons were described as the components of a virtual angular momentum (called spin)  $S = \frac{1}{2}$  in one direction,

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$M_S = \pm \frac{1}{2}$ . The common characteristics of the spin and orbital angular momentum are confirmed in more detail later. The spin state is a property of the electron itself, and the eigenfunction is not described by the wave function associated with the density distribution. Pauli proposed describing two spin states using a two-dimensional vector [2]. The eigenfunction of each spin state is as follows:

$$M_S = \frac{1}{2} \rightarrow \xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad M_S = -\frac{1}{2} \rightarrow \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (5.1.1)$$

and the general wave functions as the combination of both spin states are given by

$$\Psi = (a\xi_+ + b\xi_-) \int c(E, \vec{p}) \exp\left[\frac{2\pi i}{h}(Et + \vec{p} \cdot \vec{r})\right] dE d\vec{p}$$

$$|a|^2 + |b|^2 = 1 \quad (5.1.2)$$

The operator of the spin components in the  $x, y, z$ -directions are

$$\widetilde{S}_q = \frac{h}{4\pi} \sigma_q \quad q = x, y, z$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (5.1.3)$$

where  $\sigma_q$  are the Pauli matrices. For the Pauli matrices,

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x \sigma_y + \sigma_y \sigma_x = \sigma_x \sigma_z + \sigma_z \sigma_x = \sigma_y \sigma_z + \sigma_z \sigma_y = 0$$

$$\sigma_x \sigma_y - \sigma_y \sigma_x = 2i\sigma_z, \quad \sigma_y \sigma_z - \sigma_z \sigma_y = 2i\sigma_x, \quad \sigma_z \sigma_x - \sigma_x \sigma_z = 2i\sigma_y \quad (5.1.4)$$

are satisfied. The exchange law:

$$\widetilde{S}_x \widetilde{S}_y - \widetilde{S}_y \widetilde{S}_x = \frac{h}{2\pi} i \widetilde{S}_z, \quad \widetilde{S}_y \widetilde{S}_z - \widetilde{S}_z \widetilde{S}_y = \frac{h}{2\pi} i \widetilde{S}_x, \quad \widetilde{S}_z \widetilde{S}_x - \widetilde{S}_x \widetilde{S}_z = \frac{h}{2\pi} i \widetilde{S}_y \quad (5.1.5)$$

is the same as that for the orbital angular momentum, as shown in eq. (4.4.6). The square of the absolute value is given by:

$$(\widetilde{S}_x)^2 + (\widetilde{S}_y)^2 + (\widetilde{S}_z)^2 = \left(\frac{h}{2\pi}\right)^2 \frac{3}{4} I = \left(\frac{h}{2\pi}\right)^2 S(S+1)I \quad (5.1.6)$$

Which corresponds to the square of the absolute value of the orbital angular momentum is given by  $\left(\frac{h}{2\pi}\right)^2 L(L+1)$  as shown by eq. (4.4.12).

Using the operators

$$\widetilde{S}_+ = \widetilde{S}_x + i\widetilde{S}_y = \frac{h}{2\pi} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \widetilde{S}_- = \widetilde{S}_x - i\widetilde{S}_y = \frac{h}{2\pi} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (5.1.7)$$

the following relations hold

$$\widetilde{S}_+ \xi_+ = 0, \quad \widetilde{S}_+ \xi_- = \frac{h}{2\pi} \xi_+, \quad \widetilde{S}_- \xi_+ = \frac{h}{2\pi} \xi_-, \quad \widetilde{S}_- \xi_- = 0 \quad (5.1.8)$$

which corresponds to the relationship with the orbital angular momentum as follows (see eq. (4.4.13)):

$$\widetilde{L}_\pm \Theta_M = \frac{h}{2\pi} \sqrt{L(L+1) - M(M \pm 1)} \Theta_{M \pm 1} \quad (5.1.9)$$

Therefore, the electron spin satisfies all the relations with the angular momentum, except that the quantum numbers are given as half integers.

As shown in eq. (4.5.13), the Zeeman energy shift induced by the electron spin and the magnetic field  $B$  is obtained as follows:

$$E_Z = g_S \mu_B M_S B \quad \mu_B = \frac{eh}{4\pi m_e}$$

$$g_S = 2.002319 \quad \mu_B: \text{Bohr magneton} \quad g_S: g\text{-factor of the electron spin} \\ e: \text{unit charge} \quad m_e: \text{electron mass} \quad (5.1.10)$$

Electron spin is not derived from the Schroedinger equation treatment of the wave function as a scalar, although the Zeeman energy shift is treated as an additional perturbation term. Given that the operators of electron spin are matrices, the Hamiltonian should be a matrix. The Zeeman energy shift induced by the electron spin is derived in chapter 5.3, taking  $g_S = 2$ .

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## Fundamentals of Statistical Mechanics Using the Boltzmann Distribution

**Abstract:** The objective of this chapter is an examination of the energy distribution of matter for the highest probability, considering that only phenomena with the highest possibility are possible for a large number of masses (atoms or molecules in a macroscopic object). The thermodynamic temperature  $T$  is the parameter for the broadening of the energy distribution and the population in a state with an energy of  $E$  is proportional to  $\Omega \exp(-E/(k_B T))$ , where  $\Omega$  is the number of states and  $k_B$  is the Boltzmann constant. The average energy and specific heat are discussed using the Boltzmann distribution. The relationship between the gas pressure, volume, and temperature (ideal gas law) is obtained from the average of the one-dimensional kinetic energy. The work efficiency of the Carnot engine, using gas pressure, is also discussed.

**Keywords:** Adiabatic expansion, Bose-Einstein condensation, Carnot cycle, Entropy, Fermi degeneracy, Free energy, Ideal gas law, Specific heat, Temperature, Thermal energy, Thermal equilibrium, Vapor pressure.

### 6.1. THERMAL ENERGY

What is thermal energy? From a microscopic perspective, the constituent atoms or molecules of an object have kinetic and potential energies. However, at the macroscopic scale, the kinetic energy of the object is given by that of the motion of the center of mass (see chapter 2.2). The relative motion between the constituent atoms or molecules provides an additional energy term called thermal energy. For example, we consider the kinetic energy of gaseous molecules with a mass of  $m$ . The total energy is given by

$$E_{tot} = \frac{m}{2} \sum_{i=1}^{N_g} v_i^2 \quad N_g: \text{number of molecules} \quad (6.1.1)$$

Considering the average velocity

$$v_{ave} = \frac{\sum_{i=1}^{N_g} v_i}{N} \quad v_i = v_{ave} + \delta v_i$$

$$E_{tot} = E_K + E_T \quad (\text{note } \sum_{i=1}^{N_g} v_{ave} (\delta v_i) = 0)$$

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$$E_K = \frac{N_g m}{2} v_{ave}^2 \quad E_T = \frac{m}{2} \sum_{i=1}^{N_g} (\delta v_i)^2 \quad (6.1.2)$$

is obtained.  $E_K$  is the macroscopic kinetic energy and  $E_T$  is the thermal energy.

The macroscopic kinetic energy tends to transform into thermal energy. When an object slides on a floor, it is decelerated by friction, and the kinetic energy is transformed into thermal energy, which expands into the entire floor. Why does inverse energy flow (the matter on the floor start to move to absorb the thermal energy from the floor) not occur? Statistic mechanics dictates that the probability of the expansion of thermal energy is much higher than that of focusing on one location. When two bodies with different temperatures contact each other, the energy (heat) flows from the body at a higher temperature to another body at a lower temperature. Finally, the temperatures of both bodies become equal, for which the probability is maximum.

## 6.2. BOLTZMANN DISTRIBUTION

We consider two areas A and B with energies  $E_A$  and  $E_B$ , for which  $E_{tot} = E_A + E_B$  is constant. Considering the number of states  $\Omega_A(E_A)$  and  $\Omega_B(E_B)$  at both areas, the probability of achieving this energy distribution is proportional to the total number of states given by:

$$\Omega_{tot} = \Omega_A(E_A)\Omega_B(E_B) \quad (6.2.1)$$

When  $\Omega_{tot}$  is a maximum, the following relation holds.

$$\begin{aligned} \frac{d\Omega_{tot}}{dE_A} &= \Omega_B(E_B) \frac{d\Omega_A(E_A)}{dE_A} + \Omega_A(E_A) \frac{d\Omega_B(E_B)}{dE_A} = \Omega_B(E_B) \frac{d\Omega_A(E_A)}{dE_A} - \Omega_A(E_A) \frac{d\Omega_B(E_B)}{dE_B} = 0 \\ \frac{1}{\Omega_A(E_A)} \frac{d\Omega_A(E_A)}{dE_A} &= \frac{1}{\Omega_B(E_B)} \frac{d\Omega_B(E_B)}{dE_B} = \beta \end{aligned} \quad (6.2.2)$$

Equation (6.2.2) shows that the probability is a maximum when the parameter  $\beta$ , defined in A and B independently, is equal. Note also that the dimension  $\beta$  is the inverse of that of energy. We know empirically that the temperatures of the two contacting bodies become equal. In statistical mechanics, the thermodynamic temperature  $T$  is defined as:

$$\frac{1}{k_B T} = \beta = \frac{1}{\Omega(E)} \frac{d\Omega(E)}{dE} \quad (6.2.3)$$

Here,  $k_B$  is the Boltzmann constant, defined as  $1.38064852 \times 10^{-23}$  J/K in 2019 [1]. The statistical mechanical entropy is defined as:

$$S = k_B \ln(\Omega) \quad (6.2.4)$$

and eqs. (6.2.1-2) are rewritten as:

$$S_{tot} = S_A + S_B \quad (6.2.5)$$

$$\frac{1}{T} = \frac{dS}{dE} \quad (6.2.6)$$

We can estimate  $\Omega_B$  using  $\beta$  and  $E_A$  as follows:

$$\begin{aligned} \frac{1}{\Omega_B(E_B)} \frac{d\Omega_B(E_B)}{dE_B} &= -\frac{1}{\Omega_B(E_B)} \frac{d\Omega_B(E_B)}{dE_A} = \beta \\ \int \frac{1}{\Omega_B} \frac{d\Omega_B}{dE_A} dE_A &= -\beta \int dE_A \\ \ln(\Omega_B) &= -\beta E_A + const \\ \Omega_B &\propto \exp(-\beta E_A) = \exp\left(-\frac{E_A}{k_B T}\right) \end{aligned} \quad (6.2.7)$$

Then eq. (6.2.1) is rewritten as:

$$\Omega_{tot} \propto \Omega_A(E_A) \exp\left(-\frac{E_A}{k_B T}\right) \quad (6.2.8)$$

This energy distribution is called the ‘‘Boltzmann distribution’’. For example, the gravitational potential distribution is proportional to  $\exp\left(-\frac{mgh_g}{k_B T}\right)$  ( $g$ : gravitational acceleration,  $h_g$ : height from the ground) and the mass can be distributed within the region:

$$0 \leq h_g < \frac{k_B T}{mg} \quad (6.2.9)$$

For an object with a mass of 1 g ( $10^{-3}$  kg), the possible floating height is  $4 \times 10^{-19}$  m, which is much less than the size of a proton, and floating is impossible. However, an oxygen ( $O_2$ ) molecule ( $5.3 \times 10^{-26}$  kg) is distributed between  $h_g = 0$  and 7900 m.

## Analysis of the Measurement Uncertainties

**Abstract:** Physical laws, which have finite uncertainties, are established to facilitate measurements. New physical phenomena have been discovered when the measurement uncertainties were reduced. The object of this chapter is to review the estimation of the measurement uncertainties. This parameter consists of statistical uncertainty and systematic uncertainty. Statistical uncertainty is given by the random distribution of measurement results in a region with a broadening of  $\sigma$  in the vicinity of a real value. The statistical uncertainty of the average of the measurement results is expected to be  $\frac{\sigma}{\sqrt{N}}$ , where  $N$  is the number of measurement samples. Systematic uncertainty exists because measurements are influenced by the conditions under which they are obtained. The real value is defined for a certain circumstance, and the measurements obtained under different circumstances are shifted from this value. The real value is obtained by correcting the shift, which is estimated according to the measurement circumstances. Statistical uncertainty is obtained from the uncertainty in the estimation of the shifts.

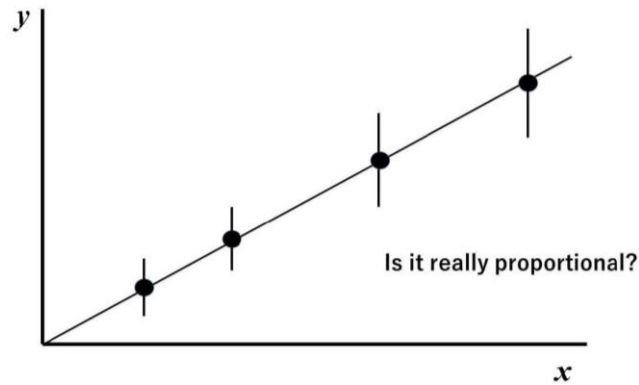
**Keywords:** Atomic clock, Central limit theorem, CPT-symmetry, Gravitational potential in the micro-scale, Measurement uncertainty, Ptolemaic and Copernican systems, Spectrum broadening, Stark shift, Statistic uncertainty, Systematic uncertainty, Zeeman shift.

### 7.1. IMPORTANCE OF THE MEASUREMENT UNCERTAINTY [1]

Physics involves the study of the laws of nature, from which we can make predictions regarding future phenomena. This is based on measurement results. For example, for the measurement results of  $(x,y) = (1,1), (2,2), (3,3), (5,5)$ , the law of  $y = x$  is established. However, uncertainty is always present in measurement results. If the measurement uncertainty is 10%, the proportionality cannot be established. The measurement result for  $x > 20$  may have a discrepancy with the established law  $y = x$  (Fig. 7.1).

Physical laws were established based on the measurement results obtained at a given time. The development of new physical laws is necessary to account for the discovery of new phenomena that are not consistent with previous physical laws. New phenomena have often been discovered *via* the minimization of measurement uncertainties. If the uncertainty of measurements are reduced to 0.5% and  $(x,y) =$

(1,0.99), (2,1.98), (3,2.95), (5,4.79) are obtained,  $y = 10\sin(x/10)$  is more appropriate description than  $y = x$ .

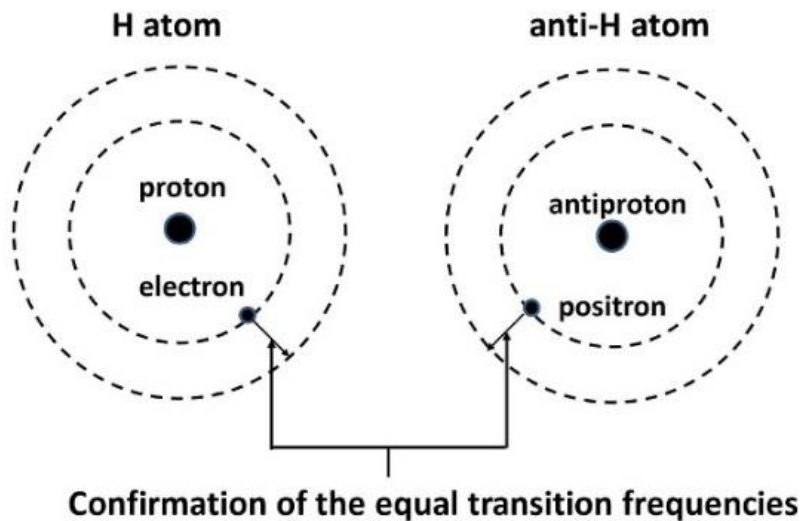


**Fig. (7.1).** All measurements have some degree of uncertainty; therefore, laws from measurement results should always be subjected. (This figure is used also in “Measurement, Uncertainty and Lasers” by M. Kajita).

The development of physics has been closely correlated to improving the accuracy of time and frequency, especially with respect to the invention of new clocks [2]. In ancient times, the Ptolemaic system was accepted because there was no discrepancy related to measurement uncertainties when it was adopted. As the accuracy of the clock improved, discrepancies were discovered regarding the position of stars at certain moments, and the Copernican system was established. Newtonian mechanics was established within 100 years after the accuracy of the clock was drastically improved by the discovery of the periodicity of the pendulum. As the accuracy of clocks continued to improve, the fluctuation of the orbital period of Io (satellite of Jupiter) was observed to be out of the margin of error of these instruments. To interpret this phenomenon, the speed of light was recognized as being finite for the first time; the light propagation time is not constant because the distance between the earth and Jupiter changes owing to their orbital motions. The characteristics of light were further elucidated because the propagation of electromagnetic waves, as described by Maxwell’s equation, is in good agreement with the measured speed of light.

Measurements with ultra-low uncertainty have been performed for atomic transition frequencies. Atomic clocks are based on the atomic transition frequencies. Fractional measurement uncertainties below  $10^{-17}$  have been obtained for several transition frequencies [3-7].

The slight shift in the transition frequencies significantly contributed to the development of modern physics. For example, time slows down in a moving frame, as shown in chapter 3.6. However, for a velocity of 1000 m/s, this effect results in a fractional shift of  $10^{-11}$ , which can only be detected using atomic clocks. The theory of general relativity shows that time slows down in the presence of a strong gravitational field (gravitational redshift). This effect results in a fractional shift of  $10^{-16}$  with a change in altitude of 1 m. For a measurement uncertainty below  $10^{-17}$ , this effect is detected when there is a change in altitude of 15 cm [3].



**Fig. (7.2).** Comparison of transition frequencies of an H-atom and an anti-H atom. (This figure is used also in “Measurement, Uncertainty and Lasers” by M. Kajita).

Further developments in physics are expected based on the detection of slight effects, for which measurements with ultralow uncertainties are required. For example, there is a mystery concerning the relationship between particles and antiparticles. Antiparticles are expected to be charge-conjugated and mirror-image inverted particles (CP-symmetry). However, why aren't antiparticles present in nature? If the CP-symmetry is violated, the number of particles can be larger with a ratio of  $10^{-9}$  compared to antiparticles, and particles can continue to exist after their antiparticles disappear *via* pair annihilation. A violation of CP-symmetry was discovered in 1964 [8]. However, CPT symmetry (antiparticles must have the image of particles after charge-conjugation, mirror-image inversion, and time-reversal) is required to maintain the Lorentz invariance; all physical laws in a coordinate must

## Conclusion

In this book, I summarize the fundamental part of analysis for the whole field of physics. Until 19<sup>th</sup> century, the fundamental of whole physics was classical mechanics, giving the equation of motion under a certain potential field. The electromagnetism gives the formula of the potential field, given by the electric and magnetic fields.

In the 20<sup>th</sup> century, the limit of validity of classical mechanics was found. Quantum mechanics is required to describe the phenomena on the micro-scale. The fundamental of quantum mechanics is wave-particle duality. The wave characteristics of particles were derived from the analogy of the characteristics of light, which were derived from Maxwell's equation. The quantum mechanics converges to the classical mechanics by  $h \rightarrow 0$  ( $h$ : Planck constant).

The theory of relativity is required to describe the phenomena with high velocity. The theory of relativity was based on the constant speed of light in the vacuum  $c$ , which was derived from Maxwell's equation. The theory of relativity converges to the classical mechanics with  $c \rightarrow \infty$ .

Therefore, all fields in physics are closely correlated. It is most important to understand the role of each field to establish the whole physics. For example, knowledge of classical mechanics is required to understand quantum mechanics. On the other hand, a simple imagination of quantum mechanics makes it easy to learn classical mechanics more in detail.

I am now writing while the world is in panic with the outbreak of coronavirus. But seeing that Newtonian mechanics was established during the plague outbreak, we might consider it as the chance to find our next step.

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Masatoshi Kajita

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## **Masatoshi Kajita**

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