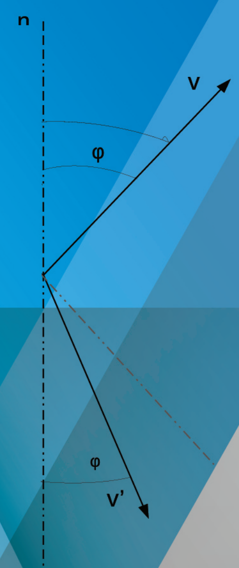
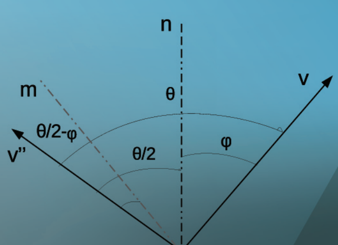
EXTERIOR CALCULUS THEORY AND CASES





$$dw_1 = d(P \wedge dx) + d(Q \wedge dy)$$

$$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right) \wedge dy$$

$$= \left(\frac{\partial P}{\partial x}\right)dx \wedge dx + \left(\frac{\partial P}{\partial y}\right)dy \wedge dx + \left(\frac{\partial P}{\partial z}\right)dz \wedge dx$$
$$+ \left(\frac{\partial Q}{\partial x}\right)dx \wedge dy + \left(\frac{\partial Q}{\partial y}\right)dy \wedge dy + \left(\frac{\partial Q}{\partial z}\right)dz \wedge dy$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy + \frac{\partial P}{\partial z}dzdx - \frac{\partial Q}{\partial z}dydz$$

Carlos Polanco

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Exterior Calculus: Theory and Cases

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Exterior Calculus: Theory and Cases

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FOREWORD

I congratulate Carlos Polanco for his experienced and insightful book Exterior Calculus – Theory and Cases. This work covers profoundly advanced Calculus for a readership that has acquired the necessary mathematical comprehension to direct to geometric algebra. It will guide higher level students as well as their teachers straightforwardly through this topic from Heaviside-Gibbs algebra over Grassmann algebra to differentiation, integration and fundamental theorems of Calculus. Despite the complexity of the subject, this book is written in a highly didactic style, which is reflecting the expertise and the long-term teaching experience of the author at the Universidad Nacional Autónoma de México.

The presentation of many examples and case studies as well the solution guide to the chapter exercises at the end of this book will help the readers to deepen and to inspect their acquired knowledge and to relate the theory to practice. I wish that Carlos Polanco's book will become part of many bookshelves and highly recommend it as a solid and distinctive textbook for advanced courses in Calculus.

Thomas Buhse
Universidad Autónoma del Estado de Morelos
Cuernavaca Morelos, Mexico.

PREFACE

This Exterior Calculus ebook has been designed for third-year students of Sciences, as it contains the fundamentals related to Geometric algebra or Grassmann algebra oriented to Calculus. Without any doubt, this algebra has important implications in Science and Engineering. Here, the reader will find a clear presentation of the Geometric algebra on a plane and in space, as well as the extension of all its operators in $\mathbb{R}^n$. In order to make the comprehension of this important algebra easier, some examples and completely solved exercises are included.

The ebook thoroughly examines the elements of Geometric algebra G over the Real **field** and these operators: inner product, outer product, and geometric product, their components, and their geometric representation, as well as their properties and the rigid transformations on the plane and in space. It also reviews the **differentiation** and the **integration** over Geometric algebra, including the line integral and surface integral. The Green, Stokes and Gauss theorems are also studied in detail and the Theorem of Fundamental Calculus is generalized.

The author hopes the reader interested in the study of the fundamentals of Exterior calculus, finds useful the material presented here and that the students that start studying this field find this information motivating. The author would like to acknowledge the Faculty of Sciences at Universidad Nacional Autónoma de México for support.

CONFLICT OF INTEREST

The author declares no conflict of interest regarding the contents of each of the chapters of this ebook.

CONSENT FOR PUBLICATION

Not applicable.

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I would like to thank all those whose recommendations made possible the publication of this ebook.

DEDICATION

The beauty of mathematics only
shows itself to more patient
followers.

Maryam Mirzakhani

List of Credits

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List of Symbols

Symbol	**Description**	**Page**
$< a >$	Blades on $\mathbb{G}_2$	28
$\|\|a\|\|$	Norm on $\mathbb{G}_2$	28
$v_{\parallel}$	Parallel component of vector v on $\mathbb{G}_2$	29
$v_{\perp}$	Perpendicular component of vector v on $\mathbb{G}_2$	29
$I = \sigma_1\sigma_2$	Pseudo-vector on $\mathbb{G}_2$	29
Ia	Clockwise rotation on $\mathbb{G}_2$	29
aI	Counter-clockwise rotation on $\mathbb{G}_2$	29
$(\mathbf{x} - x_0) \wedge v = 0$	Equation of a line on $\mathbb{G}_2$	33
$\mathbb{G}_3$	Geometric algebra on $\mathbb{R}^3$	37
$\mathbb{G}_3$	Grassmann algebra on $\mathbb{R}^3$	37
$a \wedge b$	Outer product on $\mathbb{G}_3$	38
$a \cdot b$	Inner product on $\mathbb{G}_3$	38
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$a(b+c)$	Distributivity of geometric product $\mathbb{G}_3$	43
$a \wedge (b+c)$	Distributivity of outer product $\mathbb{G}_3$	43
a^{-1}	Multiplicative inverse on $\mathbb{G}_3$	44
$a(bc) = (ab)c$	Associativity on $\mathbb{G}_3$	44
$a^{\dagger}$	Reversion on $\mathbb{G}_3$	45
Ia_r	Dual on $\mathbb{G}_3$	45
$< a >$	Blades on $\mathbb{G}_3$	46
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$v_{\perp}$	Perpendicular component of vector v on $\mathbb{G}_3$	47
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Ia	Clockwise rotation on $\mathbb{G}_3$	48
aI	Counter-clockwise rotation on $\mathbb{G}_3$	48
$(\mathbf{x} - x_0) \wedge v = 0$	Equation of a line on $\mathbb{G}_3$	50
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$a \cdot b$	Inner product on $\mathbb{G}_n$	57
ab	Geometric product on $\mathbb{G}_n$	57
$\sigma_1\sigma_2\cdots\sigma_n$	Multivector	58
$a(b+c)$	Distributivity of geometric product $\mathbb{G}_n$	58
$a \wedge (b+c)$	Distributivity of outer product $\mathbb{G}_n$	58
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Part I

Heaviside-Gibbs Algebra

The operators of the ’bf Heaviside-Gibbs algebra’ have a major role in Vector Calculus. The next chapter focuses on the definition of the main operators, showing its usefulness in solving problems in 2-dimensional and 3-dimensional space, and it also discusses the robustness and limitations of this algebra in n-dimensional space.

CHAPTER 1

Vector Algebra on $\mathbb{R}^2$ and $\mathbb{R}^3$

Abstract In this chapter, we introduce the main operators of Heaviside-Gibbs algebra: addition, subtraction, norm of vectors, as well as inner and cross product. From the point of view of Vector Calculus, we introduce the line and surface integrals, and the Green's, Stokes', and Gauss' Theorems. The last section discusses the extension of this algebra in n-dimensional space. The examples are in plane and space.

Keywords: cross product: $v \times w$, divergence of vector function, Gauss' Theorem, Green's Theorem, inner product: $v \cdot w$, limitations, line integral, norm: $||v||$, normed vector space, rotational of vector function, scalar multiplication: αv, Stokes' Theorem, surface integral, vector addition: $v+w$, vector Subtraction: $v-w$

1.1. Normed Vector Space: $V(F)$

The term **normed vector space** [5, 6, 7] is used to name a mathematical structure where a norm [7] is defined as the rules in a non-empty set V that meet the addition operation, **vector addition**, and the multiplication operation, **scalar multiplication**, between the elements of the set V and the elements of a **field** $\mathbb{F}$. This normed vector space has two important operations **inner product** [8] and **cross product** [8].

Definition 1.1. A **normed vector space** V over a **field** $\mathbb{F} \in \mathbb{R}^n$ is an algebraic structure where a set of elements called **vectors** $v, u, w \in V$ and a set of elements called **scalars** $\alpha, \beta \in \mathbb{F}$, together with two operations, **vector addition** and **scalar multiplication**, satisfy the next eight axioms [1, 4]:

Property 1. $u+(v+w) = (u+v)+w$
Property 2. $u+v = v+u$

Carlos Polanco

Property 3. $\exists 0 \in V$ i called the zero vector, such that $\forall v \in V, v+0=v$
Property 4. $\forall v \in V, \exists -v \in V,$ such that $v+(-v)=0$
Property 5. $\alpha, \beta \in \mathbb{F}, \alpha(\beta v) = (\alpha\beta)v$
Property 6. $1v = v$
Property 7. $\alpha(u+v) = \alpha u + \alpha v$
Property 8. $(\alpha+\beta)v = \alpha v + \beta v$

Remark 1.1. As it will be explained later in the chapter (Sect. 1.5), although the representation of the vectors can be in n-dimensional space, not all the operators act in this space [9, 10].

1.2. Basic operators

1.2.1. Vector Addition: $v+w$

There are two types of vectors, those that start anchored at the origin of the reference system **fixed vectors**, i.e. to a plane $\mathbb{R}^2$ or space $\mathbb{R}^3$, and those whose start is not anchored at the origin of the system **non-fixed vectors**.

Definition 1.2. The vector addition operation $\bigoplus : V \times V \to V$ takes two vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, and assigns a third vector expressed as $v+w \in \mathbb{R}^n$.

Example 1.1. Let two vectors v and $w \in \mathbb{R}^2$ be over the field $\mathbb{R}$, $v = (1,2)$ and $w = (3,-1)$. What is $v+w$?

Solution 1.1. If $v = (v_1, v_2)$ and $w = (w_1, w_2) \Rightarrow v+w = (v_1+w_1, v_2+w_2)$, then $v+w = (4,1)$.

Remark 1.2. The addition of two **fixed vectors** yields a **fixed vector**.

1.2.2. Vector Substraction: $v-w$

Definition 1.3. The vector substraction operation $\bigoplus : V \times V \to V$ takes two vectors $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$, and assigns a third vector expressed as $v-w \in \mathbb{R}^n$ where $v-w \neq w-v$.

Example 1.2. Let two vectors v and $w \in \mathbb{R}^3$ be over the field $\mathbb{R}$, $v = (1,2,-1)$ and $w = (3,-1,0)$. (i) What is $v-w$? (ii) What is $w-v$? (iii) Explain why $v-w \neq w-v$.

Solution 1.2. (i) If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3) \Rightarrow v-w = (v_1-w_1, v_2-w_2, v_3-w_3)$, then $v-w = (1,2,-1)-(3,-1,0) = (-2,3,-1)$. (ii) $w-v = (3,-1,0)-(1,2,-1) = (2,-3,1)$. (iii) In general $v-w \neq w-v$ since $v_1-w_1 \neq w_1-v_1$, where v_1, w_1 are elements of the field $\mathbb{F}$.

Remark 1.3. Any non-fixed vector can be expressed as the subtraction of two **fixed vectors**.

The **addition** of the vectors $v+(-w)$ is equivalent to $v-w$, so this **vector addition** is known as **vector subtraction**.

Two vectors are **equal** if there is a translation between them. In this sense, a **fixed vector** and a **non-fixed vector** can be the same vector.

1.2.3. Scalar Multiplication: αv

Definition 1.4. The scalar multiplication operation $\bigotimes : \mathbb{F} \times V \rightarrow V$ takes any vector $v \in \mathbb{R}^n$ and a scalar $\alpha \in \mathbb{R}$, and assigns a third vector $\alpha v \in \mathbb{R}^n$, i.e. $\alpha v = \alpha(v_1, v_2, \cdots, v_n) = (\alpha v_1, \alpha v_2, \cdots, \alpha v_n)$. When the scalar α multiplies vector v, the length of vector αv will increase or decrease; however, if $\alpha = -1$ the vector αv keeps its length but not its orientation, which will be opposite.

Example 1.3. Given vector $v = (-3, 4, 5) \in \mathbb{R}^3$ and scalar $\alpha = -3 \in \mathbb{R}$, what is vector αv?

Solution 1.3. $\alpha v = (-3)(-3, 4, 5) = (9, -12, -15)$.

This operation αv makes possible to increase the length of a vector (if $\alpha > 1$), decrease it (if $0 < \alpha < 1$), or change its orientation (if $\alpha < 0$).

1.2.4. Norm: $||v||$

Definition 1.5. The **norm** (Eq. 1.1) of a **fixed vector** $a \in \mathbb{R}^n$ represents the length or **distance** with respect to point 0.

$$||a|| = \sqrt{\sum_{i=1}^{n} a_i^2}, \text{ where } a \in \mathbb{R}^n. \tag{1.1}$$

The **norm** (Eq. 1.1) of a **non-fixed vector** $c \in \mathbb{R}^n$ represents the length or **distance** (Eq. 1.2) between the **fixed vectors** $a, b \in \mathbb{R}^n$.

$$||c|| = ||a - b|| = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}, \text{ where } c = a - b. \tag{1.2}$$

Example 1.4. There are two **fixed vectors** in a space $v = (3, 1, -2)$ and $w = (1, -1, 1)$. (i) What is the norm (or length) of vector v? (ii) What is the distance between the **fixed vectors** v and w?

Solution 1.4. (i) The norm of vector v is $||v|| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$. (ii) The distance is $||v - w|| = \sqrt{(3-1)^2 + (1-(-1))^2 + ((-2)-1)^2} = \sqrt{17}$

It is important to differentiate the **norm** of a vector $||a||$ from the absolute value of a scalar $|x|$. The first one is a vector, the second one is a real number.

1.2.5. Inner product: $v \cdot w$

Definition 1.6. The **inner product** is an algebraic operator that involves two vectors $a, b \in \mathbb{R}^n$ (Eq. 1.3) and the angle θ between them (Eq. 1.4).

$$a \cdot b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \tag{1.3}$$

Part II

Grassmann Algebra

Geometric algebra or **Grassmann algebra** is the central subject of this book. It has nine chapters: chapters 2 and 3 define this algebra in 2 and 3 dimensions; chapter 4 studies the extension to n dimensions; in chapters 5 and 6 we reformulate the derivative and integral operators; from chapter 7 to 9 we focus on the **Geometric algebra** applications to introduce the Green's, Stokes', and Gauss' theorems in Differential forms; finally, in chapter 10 we see the **Fundamental Calculus Theorem** in terms of **Geometric algebra** and Differential forms.

CHAPTER 2

Geometric Algebra on $\mathbb{G}_2$

Abstract This chapter is a review of **Geometric algebra** or **Grassmann algebra** on $\mathbb{G}_2$. This algebra is attributed to Hermann Grassmann [Die lineare Ausdehnungslehre, ein neuer Zweig der Mathematik 1842]. It has two main operators: **outer product** and **inner product**. Here, we will also study dot product, and geometric product, as well as their properties. We will start with the definition of Geometric algebra, its properties and most useful tools. With this background, we will define the differential forms in Chap. 5.

Keywords: Associativity: $a(bc) = (ab)c$, bivector, blades $< a >_i$, distributivity: $a(b+c)$, distributivity: $a \wedge (b+c)$, dual $Ia_r = b_{n-r}$, equation of a line, outer product, geometric algebra, geometric product, inner product, lines, multiplicative inverse: a^{-1}, norm $||a||$, reflections, reversion: $a^\dagger$, rotations

2.1. Geometric Algebra on $\mathbb{G}_2$

Definition 2.1. The **Geometric algebra** or **Grassmann algebra** [1, 9, 20] is a unitary associative algebra, in symbols $\mathbb{G}_2 = \mathbb{G}_2(\mathbb{R}^2)$. It is formed by three elements: α,**scalars**, σ_1, σ_2**vectors**, and the elements $\sigma_1 \wedge \sigma_2$ named **bivectors**, or **equivalently** $\sigma_1\sigma_2$, where $\alpha \in \mathbb{R}$. These elements will be expressed in **orthonormal** basis for convenience and they meet Eq. 2.1 for $i = j$.

$$\begin{aligned} \sigma_i\sigma_i &= 1 \\ \sigma_i\sigma_j &= -\sigma_j\sigma_i \end{aligned} \tag{2.1}$$

An arbitrary element will be Eq. 2.2.

$$v = \underbrace{v_0}_{basis\,scalar} + \underbrace{v_1\sigma_1 + v_2\sigma_2}_{basis\,vector} + \underbrace{v_{12}\sigma_1 \wedge \sigma_2}_{basis\,bivector} \text{ in } \mathbb{G}_2. \tag{2.2}$$

Remark 2.1. An equivalent would be $\sigma_i \wedge \sigma_j$, $\sigma_i\sigma_j$, and σ_{ij}.

Example 2.1. Provide some examples of elements on $\mathbb{G}_2$.

Solution 2.1. $v = 4\sigma_2 + 5\sigma_1 \wedge \sigma_2$, $v = 4 + \sigma_2 + -4\sigma_{12}$, $v = -1 + \sigma_1 - 3\sigma_2 + 7\sigma_{12}$.

2.1.1. Outer Product: $a \wedge b$

Definition 2.2. For two vectors $a = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_{12}$ and $b = b_0 + b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_{12} \in \mathbb{G}_2$ [1, 4, 8], we define

$$a \wedge b = \frac{1}{2}(ab - ba)$$

Example 2.2. Let two elements $a = (1, -1)$ and $b = (3, 2) \in \mathbb{G}_2$. (i) Obtain the outer product $a \wedge b = \frac{1}{2}(ab - ba)$. (ii) Obtain the geometric product using Def. 2.3.

Solution 2.2. (i) From Ex. 2.8 $a \wedge b = \frac{1}{2}(ab - ba) = 5\sigma_1\sigma_2$. (ii) $ab = a \cdot b + a \wedge b = 1 + 5\sigma_{12}$. So $a \wedge b = 5\sigma_{12}$.

The collinearity of two vectors implies that its **outer product** is zero, i.e. $a \wedge b = 0 \Leftrightarrow a \parallel b$.

Example 2.3. Let two collinear vectors $a = \sigma_1 + \sigma_2$ and $b = 2\sigma_1 + 2\sigma_2$. Determine the outer product.

Solution 2.3. $ab = 8$ and $ba = 8$, $a \wedge b = \frac{1}{2}(ab - ba) = 0$, so $a \parallel b$.

Example 2.4. Let the vectors $a = \sigma_1 + \sigma_{12}$ and $b = -2\sigma_1 + -3\sigma_2$. Determine the outer product.

Solution 2.4. $ab = -2 - 3\sigma_{12} + 2\sigma_2 - 3\sigma_1$ and $ba = -2 + 3\sigma_1 - 2\sigma_2 + 3\sigma_{12}$, $a \wedge b = \frac{1}{2}(ab - ba) = -2\sigma_2 + 3\sigma_1 + 3\sigma_{12}$.

2.1.2. Inner Product: $a \cdot b$

Definition 2.3. For two elements $a = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_{12}$ and $b = b_0 + b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_{12} \in \mathbb{G}_2$ [1, 4, 8, 15], we define

$$a \cdot b = \frac{1}{2}(ab + ba)$$

Example 2.5. Consider elements $a = \sigma_1 + \sigma_2, b = \sigma_1 - \sigma_2 \in \mathbb{G}_2$ [1, 4, 15]. (i) Obtain the geometric products ab and ba. (ii) Determine $a \cdot b$. (iii) Determine $a \wedge b$.

Solution 2.5. (i) $ab = -2, ba = 2\sigma_{12}$. (ii) $a \cdot b = 0$. (iii) $a \wedge b = 0$.

The perpendicularity of two vectors in $\mathbb{R}^2$ implies that the inner product is zero, i.e. $a \cdot b = 0 \Leftrightarrow a \perp b$.

Example 2.6. Let two perpendicular vectors [1, 21] $a = \sigma_1 + \sigma_2$ and $b = \sigma_1 - \sigma_2$ in $\mathbb{R}^2$. (i) Determine the inner product. (ii) Interpret geometrically the **inner product**.

Solution 2.6. (i) $ab = -2\sigma_{12}$ and $ba = 2\sigma_{12}$, $a \cdot b = \dfrac{1}{2}(ab + ba) = 0$, so $a \perp b$. (ii) See (Fig. 2.1).

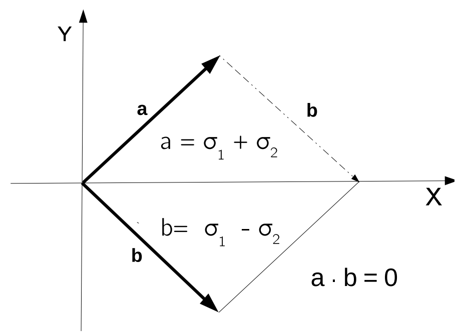

Figure 2.1 Geometrical representation of $a \cdot b$

Example 2.7. Let the vectors $a = \sigma_1 + \sigma_{12}$ and $b = -2\sigma_1 + -3\sigma_2$. Determine the inner product.

Solution 2.7. $ab = -2 - 3\sigma_{12} + 2\sigma_2 - 3\sigma_1$ and $ba = -2 + 3\sigma_1 - 2\sigma_2 + 3\sigma_{12}$, $a \cdot b = \dfrac{1}{2}(ab + ba) = -2$.

2.1.3. Geometric Product: *ab*

From these two elements $a = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_{12}$ and $b = b_0 + b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_{12} \in \mathbb{G}_2$ [1, 2, 9, 10, 16, 17, 22, 23], the **geometric product** (Eq. 2.3) is defined as

$$\begin{aligned} \mathbf{ab} &= (a_0 + a_1\sigma_1 + a_2\sigma_2 + a_{12}\sigma_1\sigma_2)(b_0 + b_1\sigma_1 + b_2\sigma_2 + b_{12}\sigma_1\sigma_2) \\ &= a \cdot b + a \wedge b \end{aligned} \tag{2.3}$$

The geometric interpretation of the **bivector** $\sigma_1 \wedge \sigma_2$ is the **oriented area** with two sides A and B spanned by the vectors σ_1 and σ_2, whose value is 1 (Fig. 2.2). Similarly, $\sigma_1 \wedge -\sigma_2$ (Fig. 2.3) represents the area of side B and $-\sigma_1 \wedge -\sigma_2$ represents the area of side A.

CHAPTER 3

Geometric Algebra on $\mathbb{G}_3$

Abstract This chapter reviews and elaborates on the operators from Geometric algebra on $\mathbb{G}_2$ to $\mathbb{G}_3$. This algebra is attributed to Hermann Grassmann [Die lineare Ausdehnungslehre, ein neuer Zweig der Mathematik 1842]. It is formed by two main operators, the **outer product** and the **inner product**, it also includes the element called **bivector**. Here, we review their properties and their application in space.

Keywords: Associativity: $a(bc) = (ab)c$, bivector: $a \wedge b$, blades $< a >$, component: $v_{\|}$, component: $v_{\perp}$, distributivity: $a(b+c)$, distributivity: $a \wedge (b+c)$, dual $Ia_r = b_{n-r}$, equation of a line, outer product, geometric algebra, geometric product, inner product, lines, multiplicative inverse: a^{-1}, norm $||a||$, reflections, reversion: $a^{\dagger}$, rotations

3.1. Geometric Algebra on $\mathbb{G}_3$

Definition 3.1. The **Geometric algebra** or **Grassmann algebra** [1, 9] is a unitary associative algebra, in symbols $\mathbb{G}_3 = \mathbb{G}_3(\mathbb{R}^3)$. It is formed by eight 2^3 elements: α, **scalars**, $\sigma_1, \sigma_2, \sigma_3$ **vectors**, $\sigma_1 \wedge \sigma_2$, $\sigma_1 \wedge \sigma_3$, $\sigma_2 \wedge \sigma_3$ **bivectors**, and $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$ **trivectors** (or **equivalently** $\sigma_1\sigma_2\sigma_3$), where $\alpha \in \mathbb{R}$. For convenience these elements are expressed in **orthonormal** basis and they meet Eqs. 3.1 for $i = j$.

$$\begin{aligned} \sigma_i\sigma_i &= 1 \\ \sigma_i\sigma_j &= -\sigma_j\sigma_i \end{aligned} \tag{3.1}$$

An arbitrary element is Eq. 3.2.

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$$\begin{aligned} v = & \underbrace{v_0}_{basis\,scalars} \\ & + \underbrace{v_1\sigma_1 + v_2\sigma_2}_{basis\,vectors} \\ & + \underbrace{v_{12}\sigma_1\wedge\sigma_2 + v_{23}\sigma_2\wedge\sigma_3 + v_{31}\sigma_3\wedge\sigma_1}_{basis\,bivectors} \\ & + \underbrace{v_{123}\sigma_1\wedge\sigma_2\wedge\sigma_3}_{basis\,trivector} \text{ in } \mathbb{G}_3. \end{aligned} \tag{3.2}$$

Remark 3.1. The equivalent would be $\sigma_i \wedge \sigma_j$, $\sigma_i\sigma_j$, and σ_{ij}.

Example 3.1. Provide some examples of elements on $\mathbb{G}_2$.

Solution 3.1. $v = 4\sigma_2 + 5\sigma_1 \wedge \sigma_2$, $v = 4 + \sigma_2 + -4\sigma_{12}$, $v = -1 + \sigma_1 - 3\sigma_2 + 7\sigma_{12}$.

3.1.1. Outer Product: $a \wedge b$

Definition 3.2. For two elements $a = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_{12}\sigma_1 \wedge \sigma_2 + a_{13}\sigma_1 \wedge \sigma_3 + a_{23}\sigma_2 \wedge \sigma_3 + \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ and $b = b_0 + b_1\sigma_1 + b_2\sigma_2 + b_{12}\sigma_1 \wedge \sigma_2 + b_{13}\sigma_1 \wedge \sigma_3 + b_{23}\sigma_2 \wedge \sigma_3 + b_{123}\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \in \mathbb{G}_3$ [1], we define

$$a \wedge b = \frac{1}{2}(ab - ba)$$

Remark 3.2. If $a \wedge b = 0 \Rightarrow a \parallel b$.

Example 3.2. Consider two elements $a = \sigma_{21} + \sigma_{123}$ and $b = 2 + \sigma_{12} \in \mathbb{G}_3$. Obtain the outer product $a \wedge b$.

Solution 3.2. Since $ab = -1 - \sigma_3 - 2\sigma_{12} + 2\sigma_{123}$ and $ba = 1 - \sigma_3 - 2\sigma 12 + 2\sigma_{123}$, $a \wedge b = 0$.

Example 3.3. Are vectores $a = \sigma_1 + \sigma_2 + \sigma_3$ and $b = 2\sigma_1 + 2\sigma_2 + 2\sigma_3$ colinear?. (i) Determine the outer product. (ii) What about Eq. 3.2.

Solution 3.3. (i) $ab = 6$ and $ba = 6$, $a \wedge b = \frac{1}{2}(ab - ba) = 0$, so $a \parallel b$. (ii) Yes, both elements are parallel.

3.1.2. Inner Product: $a \cdot b$

Definition 3.3. For two elements $a = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_{12}\sigma_1 \wedge \sigma_2 + a_{13}\sigma_1 \wedge \sigma_3 + a_{23}\sigma_2 \wedge \sigma_3 + \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ and $b = b_0 + b_1\sigma_1 + b_2\sigma_2 + b_{12}\sigma_1 \wedge \sigma_2 + b_{13}\sigma_1 \wedge \sigma_3 + b_{23}\sigma_2 \wedge \sigma_3 + b_{123}\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \in \mathbb{G}_3$ [1], we define

$$a \cdot b = \frac{1}{2}(ab + ba).$$

Remark 3.3. If $a \cdot b = 0 \Leftrightarrow a \perp b$.

Example 3.4. Let two elements $a = \sigma_1\sigma_2\sigma_3, b = \sigma_1 - \sigma_2\sigma_1\sigma_3 \in \mathbb{G}_3$. (i) Obtain the geometric products *ab* and *ba*. (ii) Determine $a \cdot b$. (iii) Determine $a \wedge b$.

Solution 3.4. (i) $ab = (\sigma_1\sigma_2\sigma_3)(\sigma_1 - \sigma_2\sigma_1\sigma_3) = \sigma_2\sigma_3 - 1$, $ba = \sigma_2\sigma_3 - 1$. (ii) $a \cdot b = \sigma_2\sigma_3 - 1$. (iii) $a \wedge b = 0$.

Example 3.5. Let two elements $a = \sigma_1\sigma_2$ and $b = \sigma_3 \in \mathbb{G}_3$. (i) Obtain the geometric products *ab* and *ba*. (ii) From Def. 3.3 determine $a \cdot b$. (iii) From Def. 3.2 determine $a \wedge b$.

Solution 3.5. (i) $ab = \sigma_1\sigma_2\sigma_3$. $ba = \sigma_1\sigma_2\sigma_3$. (ii) $a \cdot b = \sigma_1\sigma_2\sigma_3$. (iii) $a \wedge b = 0$.

Example 3.6. Let two elements $a = 1 + \sigma_1 + \sigma_2 - \sigma_2\sigma_3$ and $b = \sigma_1\sigma_2 \in \mathbb{G}_3$. (i) Obtain the geometric products *ab* and *ba*. (ii) From Def. 3.3 determine $a \cdot b$. (iii) From Def. 3.2 determine $a \wedge b$.

Solution 3.6. (i) $ab = (1 + \sigma_1 + \sigma_2 - \sigma_2\sigma_3)(\sigma_1\sigma_2) = \sigma_1 - \sigma_2 + \sigma_1\sigma_2 - \sigma_1\sigma_3$. $ba = (\sigma_1\sigma_2)(1 + \sigma_1 + \sigma_2 - \sigma_2\sigma_3) = -\sigma_1 + \sigma_2 + \sigma_1\sigma_2 - \sigma_1\sigma_3$. (ii) $a \cdot b = \sigma_1\sigma_2 - \sigma_1\sigma_3$. (iii) $a \wedge b = \sigma_1 - \sigma_2$.

Example 3.7. Let two elements a and b on $\mathbb{G}_3$ [1] $a = \sigma_1 + \sigma_2 + \sigma_3$ and $b = \sigma_1 + \sigma_2 - 2\sigma_3$ in $\mathbb{R}^3$. (i) Determine the inner product. (ii) Give a geometrical interpretation of the **inner product**.

Solution 3.7. (i) $ab = 3\sigma_{13} + 3\sigma_{23}$ and $ba = -3\sigma_{13} - 3\sigma_{23}$, $a \cdot b = \frac{1}{2}(ab + ba) = 0$, so $a \perp b$. (ii) See Fig. 3.1.

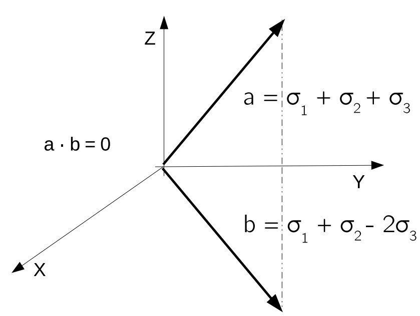

Figure 3.1 Geometrical representation of $a \cdot b$.

3.1.3. Geometric Product: *ab*

For two elements $a = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_{12}\sigma_1 \wedge \sigma_2 + a_{13}\sigma_1 \wedge \sigma_3 + a_{23}\sigma_2 \wedge \sigma_3 + \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ and $b = b_0 + b_1\sigma_1 + b_2\sigma_2 + b_{12}\sigma_1 \wedge \sigma_2 + b_{13}\sigma_1 \wedge \sigma_3 + b_{23}\sigma_2 \wedge \sigma_3 + b_{123}\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \in \mathbb{G}_3$ [1], the **geometric product** (Eq. 3.3) is defined as

CHAPTER 4

Geometric Algebra on $\mathbb{G}_n$

Abstract This chapter reviews and elaborates on the operators of Geometric algebra from $\mathbb{G}_3$ to $\mathbb{G}_n$. This algebra is attributed to Hermann Grassmann [Die lineare Ausdehnungslehre, ein neuer Zweig der Mathematik 1842]. It is formed by two main operators, the **outer product** and **inner product**. Here, a new element is introduced the **multivector**, we review these operators, their properties, and their application in the representation of curves, planes, and objects on space $\mathbb{G}_n$.

Keywords: Associativity: $a(bc) = (ab)c$, bivector: $a \wedge b$, blades $< a >$, component: $v_\parallel$, component: $v_\perp$, distributivity: $a(b+c)$, distributivity: $a \wedge (b+c)$, dual $Ia_r = b_{n-r}$, equation of a line, outer product, geometric algebra, geometric product, inner product, lines, multiplicative inverse: a^{-1}, multivector $a \wedge b \wedge c \wedge \cdots \wedge z$, norm $||a||$, reflections, reversion: $a^\dagger$, rotations, trivector: $a \wedge b \wedge c$

4.1. Preliminaries

This chapter explores the main operators in space $\mathbb{G}_n$, since this space corresponds to $\mathbb{G}_n = \mathbb{G}_n(\mathbb{R}^n)$, we will **not** provide illustrative graphs, but we will focus on the analytical solutions oriented to the elements in that space using the **outer product** $a \wedge b$.

Note 4.1. It is important to note that although the elements $\mathbb{G}_n$ have $\sigma_{1 \cdots n}$ (Def. 4.1), to simplify, we have replaced them with examples on $\mathbb{G}_4$.

4.2. Geometric Algebra on $\mathbb{G}_n$

Definition 4.1. The **Geometric algebra** or **Grassmann algebra** [1, 9] is a unitary associative algebra, in symbols $\mathbb{G}_n = \mathbb{G}_n(\mathbb{R}^n)$. It is formed by 2^n elements:

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scalars α_i, **vectors** $\alpha_i\sigma_i$, **bivectors** $\alpha_{ij}\sigma_{ij}$, **trivectors** $\alpha_{ijk}\sigma_{ijk}$, and **multivectors** $\alpha_{i\cdots n}\sigma_{i\cdots n}$. For convenience, these elements are expressed in **orthonormal** basis that meet Eqs. 4.1 for $i \neq j$.

$$\begin{aligned}\sigma_i\sigma_i &= 1\\ \sigma_i\sigma_j &= -\sigma_j\sigma_i\end{aligned} \tag{4.1}$$

An arbitrary element is (Eq. 4.2).

$$\begin{aligned} v = &\underbrace{\sum_{i=1}^{n} v_i}_{basis\ scalars} + \underbrace{\sum_{i=1}^{n} v_i\sigma_i}_{basis\ vectors} + \underbrace{\sum_{i,j=1}^{n} v_{ij}\sigma_{ij}}_{basis\ bivectors}\\ &+ \underbrace{\sum_{i,j,k=1}^{n} v_{ijk}\sigma_{ijk}}_{basis\ trivectors} + \underbrace{\sum_{i,\cdots,z=1}^{n} v_{i\cdots z}\sigma_{i\cdots z}}_{basis\ multivector} \text{ in } \mathbb{G}_n. \end{aligned} \tag{4.2}$$

Remark 4.1. The equivalent is $\sigma_i \wedge \sigma_j$, $\sigma_i\sigma_j$ and σ_{ij}.

Example 4.1. Provide some examples of elements on $\mathbb{G}_n$.

Solution 4.1. $v = 5\sigma_{1\cdots n}$,

4.2.1. Outer Product: $a \wedge b$

Definition 4.2. For two elements a and $b \in \mathbb{G}_n$ [1], we define

$$a \wedge b = \frac{1}{2}(ab - ba)$$

Remark 4.2. If $a \wedge b = 0 \Rightarrow a \parallel b$.

Example 4.2. Consider two elements $a = \sigma_{1234}$ and $b = 2 + \sigma_{12} \in \mathbb{G}_n$. Obtain the outer product $a \wedge b$.

Solution 4.2. $ab = 2\sigma_{1234} + \sigma_{123412} = 2\sigma_{1234} - \sigma_{34}$ and $ba = 2\sigma_{1234} - \sigma_{34}$, so $a \wedge b = 2\sigma_{34}$.

Example 4.3. Let two elements a and b on $\mathbb{G}_n$ $a = \sigma_5 + \sigma_1$, where $b = \alpha\sigma_5 + \beta\sigma_1$. Determine what values comply with the scalars α and β so both vectors are collinear.

Solution 4.3. If $a \wedge b = \frac{1}{2}(ab - ba) = 0$, then $a \parallel b$. Since $ab = 2\alpha$, $ba = 2\beta$ $a \wedge b = \frac{1}{2}(ab - ba) = 0 \Leftrightarrow 2\alpha - 2\beta = 0 \Leftrightarrow \alpha = \beta$.

4.2.2. Inner Product: $a \cdot b$

Definition 4.3. For two elements a and $b \in \mathbb{G}_n$ [1], we define

$$a \cdot b = \frac{1}{2}(ab + ba).$$

Remark 4.3. If $a \cdot b = 0 \Leftrightarrow a \perp b$.

Example 4.4. Let two elements $a = \sigma_{567}, b = \sigma_{12345} \in \mathbb{G}_n$. (i) Obtain the geometric products ab and ba. (ii) Determine $a \cdot b$. (iii) Determine $a \wedge b$.

Solution 4.4. (i) $ab = -\sigma_{123467}$, $ba = \sigma_{123467}$. (ii) $a \cdot b = 0$. (iii) $a \wedge b = -\sigma_{123467}$.

Example 4.5. Let two elements $a = \sigma_1\sigma_2$ and $b = \sigma_3 \in \mathbb{G}_n$. (i) Obtain the geometric products ab and ba. (ii) From Def. 4.3, determine $a \cdot b$. (iii) From Def. 4.2, determine $a \wedge b$.

Solution 4.5. (i) $ab = \sigma_1\sigma_2\sigma_3$. $ba = \sigma_1\sigma_2\sigma_3$. (ii) $a \cdot b = \sigma_1\sigma_2\sigma_3$. (iii) $a \wedge b = 0$.

Example 4.6. Let two elements $a = 1 + \sigma_1 + \sigma_2 - \sigma_{24}$ and $b = \sigma_{1234} \in \mathbb{G}_n$. (i) Obtain the geometric products ab and ba. (ii) From Def. 4.3, determine $a \cdot b$. (iii) From Def. 4.2, determine $a \wedge b$.

Solution 4.6. (i) $ab = (\sigma_{1234})(\sigma_1 + \sigma_2 - \sigma_{24}) = \sigma_{234} - \sigma_{134} - \sigma_{13}$. $ba = -\sigma_{234} + \sigma_{134} - \sigma_{13}$ (ii) $a \cdot b = -\sigma_{13}$. (iii) $a \wedge b = \sigma_{234} - \sigma_{134}$.

Example 4.7. Let two elements a and b on $\mathbb{G}_n$ [1] $a = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$ and $b = \sigma_1 + \sigma_2 - 2\sigma_3 + \sigma_4$ on $\mathbb{G}_n$. (i) Determine the inner product. (ii) Geometrically interpret the **inner product**.

Solution 4.7. (i) $ab = 1 - 2\sigma_{13} + \sigma_{23} + 3\sigma_{34}$ and $ba = 1 + 3\sigma_{13} + 2\sigma_{23} - 3\sigma_{34}$, $a \cdot b = \frac{1}{2}(ab + ba) = 1$. (ii) It is a point in the $\mathbb{G}_4$ space.

4.2.3. Geometric Product: ab

From two elements a and $b \in \mathbb{G}_n$, the **geometric product** (Eq. 4.3) is defined as [1]

$$\mathbf{ab} = a \cdot b + a \wedge b \tag{4.3}$$

where the term $\mathbf{a} \cdot \mathbf{b}$ is the **inner product** (Def. 4.2.2) and the term $\mathbf{a} \wedge \mathbf{b}$ is the **outer product** (Def. 4.2.1) [1].

Remark 4.4. If the elements on $\mathbb{G}_n$ are of the form $a = a_1\sigma_1 + a_2\sigma_2 + \cdots + a_n\sigma_3$ and $b = b_1\sigma_1 + b_2\sigma_2 + \cdots + b_n\sigma_n$, then the **inner product** will only have the **scalar part** and the **outer product** the **vectorial part**.

CHAPTER 5

Differentiation

Keywords: 0−Forms, 1−Forms, 2−Forms, 3.Forms, k−Forms, $d\eta$, df, dx_i, $dx_i \wedge dx_j$, $dx_i \wedge dx_j \wedge dx_k$, $dx_i \wedge dx_j \wedge dx_k \wedge dx_l$, $dx_i \wedge dx_j \wedge \cdots \wedge dx_n$, dw, $d\eta$, $d(w \wedge \eta)$, derivative of 0−form, derivative of 1−form, derivative of 2−form, derivative of 3−form, derivative of k−form, differential forms, divergence, exterior derivative, function w, function η, geometric product, geometric product, gradient, inner product, outer product, rotational, tangent line, tangent plane

5.1. Differential of a Function

Informally, an approximation to the definition of a differential of the function $f : \mathbb{R} \to \mathbb{R}$ is $dy = f'(x)dx$. If $dy < 0$, then $(dy)^2$ is negligible, i.e. $(dy)^2 \approx 0$ [11]. This assumption is useful to obtain the **derivative**, or exterior derivative, of an **outer product**.

If we substitute the elements σ_i in **orthonormal basis** of the **geometric product** by the **differentials** dx_i and consider $(dy)^2 \approx 0$ (Prop. 5.1), then we can define the **families** (of both real-valued functions and vector-valued functions), whose basis are formed by dx_i that act on a **tangent plane**.

This type of **function families** are known as **Differential forms**.

Remark 5.1. **The equivalent is** $dx_i \wedge dx_j$**,** $dx_i dx_j$**, and** dx_{ij}**.**

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5.2. Differential Forms

5.2.1. $0-$Forms

A $0-$form is any differentiable real-valued function $f(x)$ defined to assign a unique **real number** to a point, i.e. a 0-form is the measure of a flux over a point in an infinitesimal $0-$region [27].

Definition 5.1. A 0-form in $\mathbb{R}^n$ is a **differentiable real-valued function** w_0 (Eq. 5.1) [4].

$$w_0 = f : \mathbb{R}^n \to \mathbb{R} \tag{5.1}$$

Example 5.1. Determine the product and sum of the functions $w_{01}(x,y) = e^x + 3y$ and $w_{02}(x,y) = x - y$.

Solution 5.1. (i) $w_0(x,y) = w_{01}(x,y) + w_{02}(x,y) = e^x + 3y + x - y = e^x + 2y + x$. (ii) $w_0(x,y) = w_{01}(x,y)w_{02}(x,y) = (e^x + 3y)(x - y) = xe^x - ye^x + 3yx - 3y^2$.

Example 5.2. Determine the product and sum of the functions $w_{01}(x) = \sin x$ and $w_{02}(x) = \cos x$.

Solution 5.2. (i) $w_0(x) = w_{01}(x) + w_{02}(x) = \sin x + \cos x$. (ii) $w_0(x) = w_{01}w_{02}(x) = \sin x \cos x$.

5.2.2. $1-$Forms

A $1-$form is any differentiable vector-valued function $f(x)$ defined to assign a unique **real number** to an oriented curve, i.e. a 1-form is the measure of a flux over an oriented curve in an infinitesimal $1-$region [27].

Definition 5.2. A $1-$Form in $\mathbb{R}^n$ is a **vector-valued function** formed by a linear combination of the real-valued functions $f_i : \mathbb{R}^n \to \mathbb{R}$ over an **orthonormal basis**, formed by the differentials dx_i (Eq. 5.2) [4].

$$w_1 = f_1(x_1, \cdots, x_n)\, dx_1 + \cdots + f_n(x_1, \cdots, x_n)\, dx_n \tag{5.2}$$

Example 5.3. Determine the product and the sum of the functions $w_{11}(x,y) = e^x\, dx + 3y\, dy$, $w_{12}(x,y) = x\, dx - y\, dy$, and $w_0(x,y) = xy$.

Solution 5.3. (i) $w_{11}(x,y) + w_{12}(x,y) = (e^x + x)\, dx + (3y - y)\, dy$. (ii) $w_0(x,y)\, w_{11}(x,y) = (xy)e^x\, dx + (xy)3y\, dy = xye^x\, dx + 3xy^2\, dy$.

Example 5.4. Determine the product and the sum of the functions $w_{11}(x) = \sin x\, dx$ and $w_{12}(x) = \cos x\, dx$.

Solution 5.4. (i) $w_1(x) = w_{11}(x) + w_{12}(x) = (\sin x + \cos x)\, dx$. (ii) If $w_0(x) = \tan x$ then $w_0(x)w_{11}(x) = \tan x \sin x\, dx$.

Example 5.5. Determine the product and the sum of the functions $w_{11}(x) = \sin x\,dx$ and $w_{12}(x) = \cos x\,dy$.

Solution 5.5. (i) $w_1(x) = w_{11}(x) + w_{12}(x) = (\sin x\,dx + \cos x)\,dy$. (ii) If $w_0(x) = \tan x$ then $w_0(x)w_{11}(x) = \tan x \sin x\,dx$.

5.2.3. $2-$Forms

A $2-$form is any differentiable vector-valued function $f(x)$ defined to assign a unique **real number** to an oriented surface, i.e. a 2-form is the measure of a flux in an infinitesimal $2-$region [27].

Definition 5.3. A $2-$Form in $\mathbb{R}^n$, is a **vector-valued function** formed by a linear combination of the real-valued functions $f_i : \mathbb{R}^n \to \mathbb{R}$ over an **orthonormal basis** of the differentials $dx_i \wedge dx_j$ (Eq. 5.3) [4].

$$w_2 = f_1(x_1, \cdots, x_n)\,dx_1 \wedge dx_2 + \cdots + f_n(x_1, \cdots, x_n)\,dx_i \wedge dx_j \tag{5.3}$$

Example 5.6. Determine the product and the sum of the functions $w_{21}(x,y,z) = e^x\,dx \wedge dy + 3zy\,dx \wedge dz + \cos x\,dy \wedge dz$, $w_{22}(x,y,z) = xz\,dx \wedge dy - y\,dx \wedge dz$, and $w_0(x,y,z) = xy$.

Solution 5.6. (i) $w_{21}(x,y,z) + w_{22}(x,y,z) = (e^x + xz)\,dx \wedge dy + (3yz - y)\,dx \wedge dz + \cos x\,dy \wedge dz$. (ii) $w_0(x,y,z)w_{21}(x,y,z) = (xy)e^x\,dx \wedge dy + 3xy^2z\,dx \wedge dz + xy\cos x\,dy \wedge dz$.

Example 5.7. Determine the product and the sum of the functions $w_{21}(x,y) = \sin x\,dx \wedge dy$ and $w_{22}(x,y) = \cos x\,dx \wedge dy$.

Solution 5.7. (i) $w_1(x,y) = w_{21}(x,y) + w_{22}(x,y) = (\sin x + \cos x)\,dx \wedge dy$. (ii) If $w_0(x,y) = \tan x$ then $w_0(x,y)w_{21}(x,y) = \tan x \sin x\,dx \wedge dy$.

5.2.4. $3-$Forms

A $3-$form is any differentiable vector-valued function $f(x)$ defined to assigne a unique **real number** to an oriented volume, i.e. a 3-form is the measure of a flux over an oriented volume in an infinitesimal $3-$region, it is the measure of a fluid [27].

Definition 5.4. A $3-$Form in $\mathbb{R}^n$ is a **vector-valued function** formed by a linear combination of the real-valued functions $f_i : \mathbb{R}^n \to \mathbb{R}$ over an **orthonormal basis** of the differentials $dx_i \wedge dx_j \wedge dx_k$ (Eq. 5.4).

$$w_3 = f_1(x_1, \cdots, x_n)\,dx_1 \wedge dx_2 \wedge dx_3 + \cdots + f_n(x_1, \cdots, x_n)\,dx_i \wedge dx_j \wedge dx_k \tag{5.4}$$

Example 5.8. Determine the product and the sum of the functions $w_{31}(x,y,z) = e^x\,dx \wedge dy \wedge dz + 3zy\,dx \wedge dz \wedge dy + \cos x\,dy \wedge dz \wedge dx$, $w_{32}(x,y,z) = xz\,dx \wedge dy \wedge dz - y\,dx \wedge dz \wedge dy$, and $w_0(x,y,z) = xy$.

CHAPTER 7

Fundamental Theorem of Calculus

Abstract This chapter reviews The Green's, Stokes', and Gauss' Theorems as a direct result of the differentiation and integration operations set out in previous chapters. All the exercises are solved using the Grassmann algebra. The Fundamental theorem of calculus is introduced at the end of this chapter, as an extension of the theorems studied here.

Keywords: dw_1 form, dw_2 form, w_1 form, w_2 form, divergence, Field associated, Fundamental Theorem of Calculus, Gauss' theorem, Grassmann algebra, Green's theorem, Heaviside-Gibbs algebra, rotational, Stokes' theorem

7.1. Preliminaries

In the following sections, we will introduce the operators and properties of the Grassmann algebra (Chaps. 2-6) with some examples. These operators and their properties are required to introduce the Green's, Stokes', and Gauss' theorems. If the reader is interested in knowing these theorems under the Heaviside Gibbs algebra, he/she can review (Chap. 1).

These theorems derive directly from the integration and differentiation operators of Grassmann algebra and their generalization is provided in the last section of this chapter.

7.2. Green Theorem

Definition 7.1. Let w_1 be a $1-$form on an open over a region $D \subset \mathbb{R}^2$ bounded by ∂D in the positive perimeter, then

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$$\int_{\partial D} w_1 = \int_D dw_1 \tag{7.1}$$

Green's theorem states that the effect of the vector-valued function F over the oriented closed curve ∂D, counter-clockwise orientation (represented by w_1-form over $\mathbb{R}^2$), is equivalent to the rotational effect over the area bounded by the region D, i.e. dw_1.

Proof. The definition of field associated (Sect. 5.3.2) is verified.

Example 7.1. Let $w_1 = -y\,dx + x\,dy$. Verify Green's theorem over the region $c(t) = (\cos t, \sin t), t \in [02\pi]$.

Solution 7.1. $\int_{\partial D} w_1 = \int_0^{2\pi} -y\,dx + x\,dy\,dt = \int_0^{2\pi} -\sin t(\cos t)_t' + \cos t(\sin t)_t'\,dt$
$= \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt = 2\pi.$

$$\begin{aligned}
dw_1 &= d(-y \wedge dx) + d(x \wedge dy)) \\
&= -\left(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial y}dy\right) \wedge dx + \left(\frac{\partial x}{\partial x}dx + \frac{\partial x}{\partial y}dy\right) \wedge dy \\
&= -\left(\frac{\partial y}{\partial x}\right) dx \wedge dx - \left(\frac{\partial y}{\partial y}\right) dy \wedge dx \\
&+ \left(\frac{\partial x}{\partial x}\right) dx \wedge dy + \left(\frac{\partial x}{\partial y}\right) dy \wedge dy \\
&= 2dxdy
\end{aligned} \tag{7.2}$$

now, we parameterize $c(r,\theta) = (r\cos\theta, r\sin\theta, 1)$

$\int_D dw_1 = \int_0^1 \int_0^{2\pi} [2dxdy]\,d\theta dr = \int_0^1 \int_0^{2\pi} \left[2\frac{\partial(x,y)}{\partial(r,\theta)}\right] d\theta dr = \int_0^1 \int_0^{2\pi} 2r\,d\theta dr$
$= 2\pi.$

Note 7.1. $\frac{\partial(x,y)}{\partial(r,\theta)} = r.$

The Green's theorem is verified.

7.3. Stokes' Theorem

Definition 7.2. Let w_1 be a $1-$form on an open over a region $S \subset \mathbb{R}^3$ bounded by ∂S in the positive perimeter, then

$$\int_{\partial S} w_1 = \int_S dw_1 \tag{7.3}$$

Stokes's theorem states that the effect of the vector-valued function F over the oriented closed curve ∂D, counter-clockwise orientation (represented by w_1-form over $\mathbb{R}^3$), is equivalent to the rotational effect over the area bounded by the region D, i.e. dw_1.

Proof. The definition of field associated (Sect. 5.3.2) is verified.

Example 7.2. Let $w_1 = xy\,dx + e^z\,dy + x\,dz$. Verify Stokes' theorem over the region $c(t) = (\cos t, \sin t, 1)$.

Solution 7.2. $\int_{\partial D} w_1 = \int_0^{2\pi} xy\,dx + e^z\,dy + x\,dz\,dt = \int_0^{2\pi} \cos t \sin t (\cos t)_t' + e^1$ $(\sin t)_t' + \cos t(1)_t'\,dt = \int_0^{2\pi} -\cos t \sin^2 t + e^t \cos t\,dt = 0.$

$$\begin{aligned}
dw_1 &= d(xy \wedge dx) + d(e^z \wedge dy) + d(x \wedge dz) \\
&= \left(\frac{\partial xy}{\partial x}dx + \frac{\partial xy}{\partial y}dy + \frac{\partial xy}{\partial z}dz\right) \wedge dx \\
&+ \left(\frac{\partial e^z}{\partial x}dx + \frac{\partial e^z}{\partial y}dy + \frac{\partial e^z}{\partial z}dz\right) \wedge dy \\
&+ \left(\frac{\partial x}{\partial x}dx + \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz\right) \wedge dz \\
&= \left(\frac{\partial xy}{\partial x}\right)dx \wedge dx + \left(\frac{\partial xy}{\partial y}\right)dy \wedge dx + \left(\frac{\partial xy}{\partial z}\right)dz \wedge dx \\
&+ \left(\frac{\partial e^z}{\partial x}\right)dx \wedge dy + \left(\frac{\partial e^z}{\partial y}\right)dy \wedge dy + \left(\frac{\partial e^z}{\partial z}\right)dz \wedge dy \\
&= \left(\frac{\partial x}{\partial x}\right)dx \wedge dz + \left(\frac{\partial x}{\partial y}\right)dy \wedge dz + \left(\frac{\partial x}{\partial z}\right)dz \wedge dz \\
&= -xdxdy + dzdx + e^z dzdy
\end{aligned} \tag{7.4}$$

now, we parameterize $c(r,\theta) = (r\cos\theta, r\sin\theta, 1)$

$\int_D dw_1 = \int_0^1 \int_0^{2\pi} [-xdxdy + dzdx + e^z dzdy]\,d\theta dr = \int_0^1 \int_0^{2\pi} \Big[-r\cos\theta \frac{\partial(x,y)}{\partial(r,\theta)}$ $+ \frac{\partial(z,x)}{\partial(r,\theta)} + e^1 \frac{\partial(z,y)}{\partial(r,\theta)}\Big] d\theta dr = \int_0^1 \int_0^{2\pi} -r^2 \cos\theta\,d\theta dr = 0.$

Note 7.2. $\frac{\partial(x,y)}{\partial(r,\theta)} = r$, $\frac{\partial(z,x)}{\partial(r,\theta)} = 0$, and $\frac{\partial(z,y)}{\partial(r,\theta)} = 0$.

The Stokes' theorem is verified.

7.4. Gauss' Theorem

Definition 7.3. Let w_2 be a $2-$form on an open over a region $\omega \subset \mathbb{R}^3$ bounded by $\partial\omega$, in the positive perimeter, then

$$\iint_{\partial\omega} w_2 = \iiint_{\omega} dw_2 \tag{7.5}$$

III

APPLICATIONS

The second part of the book starts with the characterization of the real-valued functions with a review of the concepts of continuity, differentiation, and integration. Integration is presented with mappings on a plane and space. We define the vector-valued functions, their geometric representation and the two vector operators: rotational and divergence. Each section is self-contained so the unfamiliar reader can follow up the subject.

CHAPTER 8

Applications

Abstract This chapter gives an alternative solution to the spread of an epidemic outbreak of k dimension, using a $k-$Form. The $k-$region, derivative, and integral of this $k-$Form are interpreted. An extension of the k dimension is proposed using a $k-$Form equivalent to the electric current and the magnetic field, known as Ampere's law. An algorithm to determine the main function of a protein is introduced using a $k-$Form. Finally, the $k-$region, derivative, and integral of this $k-$Form are interpreted.

Keywords: Ampere's law, clinical variables, mathematical epidemiology, non-clinical variables, structural proteomics

8.1. Mathematical Epidemiology

8.1.1. Preliminaries

In recent years, after the emblematic analysis of 335 infectious emerging diseases from 1940 to 2004, in which it was reported that 60% were zoonosis and 25% were mosquito-borne viruses [30], and after the A-H1N1 flu outbreak of 1989 [31], there has been substantial progress in the development of surveillance systems of serious diseases with epidemic potential to support public health, clinical infrastructure, and the limited responsiveness of Emergency Services.

At present, it is still uncertain if a sporadic zoonosis restricted to a certain area will become a global pandemic or something in between. Therefore, surveillance systems of severe infectious diseases with epidemic potential should not only be based on the number of notified cases and their space-time distribution in a determined geographical area, to issue an early warning.

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The best would be to also consider non-clinical variables, such as socio-demographic factors, public transport, livestock production, and vaccinated population, as it is known [31] that these factors are the epidemiological foundation for the spread of a potential pandemic outbreak. Today, a person can be infected on one continent and be on another 10 hours later.

This, combined with the virulence of the pathogenic agents and some socio-demographic factors, determine their spreading capability. A surveillance system of severe infection diseases with epidemic potential, will give health authorities valuable time to promote suitable measures and minimize the spread of the disease.

For this reason, it is expected that a surveillance system of severe infectious diseases with epidemic potential identifies, as soon as possible, specific symptomatic cases of an infectious process; this requires a predictive element that foresees, with a certain degree of accuracy, a possible event in the time/space of this infectious process so the authorities take preventive measures in the affected region.

In our opinion, two of the main factors undermining the effectiveness of the warnings are, on the one hand, the increasingly efficient means of transport and on the other, the numerous mild diseases e.g. colds that present fever.

Nowadays, the surveillance systems of serious infectious diseases with epidemic potential are mainly based on the number of microbiologically [32] verified cases; the warnings, although real, are also late as monitoring is based on the assumption that symptomatic subjects will go to a clinic.

However, if the transmissibility and/or lethality of the virus is very high, or if the number of medical facilities in the area is very limited, the index patient and some of his/her contacts will probably die before receiving medical attention, which will make even harder to trace back the contacts net that will continue growing. Additionally, the number of doctors and clinics available is frequently less than optimal, as in the case of developing countries, where the population does not usually seek medical advice for many different reasons.

In this circumstance, it is necessary to have a predictive model of serious infectious diseases with pandemic potential that considers and weights clinical and non-clinical variables, instead of depending only on the number of microbiologically confirmed cases, and that forecasts the emergence and progress of the outbreak in a region.

8.1.2. Model

The model proposed, defines a $k-$form function (Eq. 5.20), whose dx_k is a measure of the net flux through the boundary of an infinitesimal $(k+1)-$region, enclosed in an oriented $k-$geographical region that represents the effect of the total flux on a particular area of that $k-$geographical region,
where

Definition 8.1. The derivative [3] of a $3-$form function w_k (Eq. 8.1) is a $k+1-$form function of C^1 class $w_{k+1} = dw_k$ (Eq. 8.2).

$$w_k = f(x_1, \cdots, x_n)\, dx_j \wedge \cdots \wedge dx_k \tag{8.1}$$

$$dw_k = \left(\sum_{i=1}^{n} \frac{\partial f(x_1, \cdots, x_n)}{\partial x_i}\, dx_i \right) \wedge dx_j \wedge \cdots \wedge dx_k \tag{8.2}$$

and,

Definition 8.2. The integral of a differential form w_k in $\mathbb{R}^n$ over a region $D \in \mathbb{R}^n$, is represented by (Eq. 8.3).

$$\int_D w_k = \int_{a_1}^{a_k} \cdots \int_{a_{k+m}}^{a_n} f_1(x_1, \cdots, x_n)\, dx_1 \cdots dx_2 + \cdots + f_n(x_1, \cdots, x_n)\, dx_n \cdots dx_1\, dD \tag{8.3}$$

The previous definitions are in (Chaps. 5, and 6) if the reader wants to deepen in these concepts, it is advisable to review these chapters.

8.1.2.1 Clinical Variables

Clinical variables [31] are parameters strongly associated with an epidemic process and they are related to the seriousness of the patients' condition, or the medical supplies necessary for their attention, i.e. hemodynamic monitors and mechanical ventilators.

8.1.2.2 Non-Clinical Variables

Non-clinical variables associated with an epidemic process are those variables that are not associated with the medical aspect and may well be associated with transport phenomena, education, population growth, or accessibility to drinking water, e.g. passengers traveling, illiterate indigenous population, immigrant population, and dwellings without piped water.

8.1.3. Algorithm

The function is a vector-valued function $f : \mathbb{R}^k \to \mathbb{R}^k$, where k is the number of clinical and non-clinical variables.

The integral 6.6 of a $k-$form represents the total effect or flux that a vector-valued function $f(x)$ has over the oriented $k-$volume, on an interval of the domain of the function; and the derivative (Def. 5.3.5) dx_k is a measure of the net flux through the boundary of an infinitesimal $(k+1)-$region enclosed in an oriented $k-$volume.

CHAPTER 9

SOLUTIONS

Solutions Chapter 1

Solution 1.1. (i) The map is $T : \mathbb{R} \Rightarrow \mathbb{R}^2, (a\cos\theta, b\sin\theta)$, $\theta \in [0, 4\pi]$. (ii) See (Fig. 1.2).

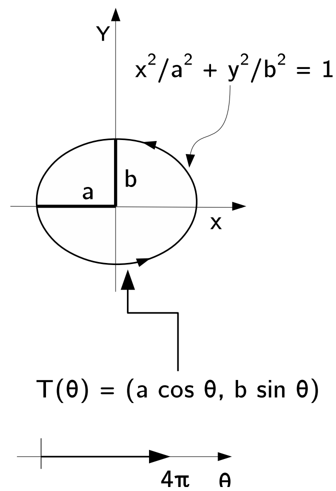

Figure 1.2 Map of the ellipse where $b < a$. Figure adapted from [1].

(iii) The mapping runs twice the perimeter of the ellipse.

Solution 1.2. The map is $T : \mathbb{R} \Rightarrow \mathbb{R}^3, (\cos\theta, \sin\theta, 1 - \cos\theta - \sin^3\theta)$, $\theta \in [0, 3\pi]$.

Solution 1.3. (i) The paraboloid is the graph of the function $f(x,y) = 1 - x^2 - y^2$, the map $T : \mathbb{R}^2 \Rightarrow \mathbb{R}^2, (r\cos\theta, r\sin\theta)$, $r \in [0,1], \theta \in [0, 2\pi]$ transforms the rectangle into the unit circle and $f \circ T$ is the third component of the map $T : \mathbb{R}^2 \Rightarrow \mathbb{R}^3$. Then, $T(r,\theta) = (r\cos\theta, r\sin\theta, 1 - r^2\cos^2\theta - r^2\sin^2\theta) = (r\cos\theta, r\sin\theta, 1 - r^2)$.

Solution 1.4. (i) $\oint_C F \circ c(t) \cdot c'(t)\, dt = \int_0^{2\pi} (-\sin^2 t, \cos t) \cdot (-\sin t, \cos t) dt = \pi$.

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Solution 1.5. (i) $\oint_C F\circ c(t)\cdot c'(t)\,dt = \int_0^{2\pi}(-\sin^2 t,\cos t,1)\cdot(-\sin t,\cos t,0)dt = \pi$.

Solution 1.6. (i) $T(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{1-r^2})$. $\oiint_S F\circ T(u,v)\cdot\eta(u,v)dS = \int_0^{2\pi}\int_0^1(r\sin\theta, r\cos\theta,\sqrt{1-r^2})\cdot(\frac{-r^2\cos\theta}{\sqrt{1-r^2}},\frac{r^2\sin\theta}{\sqrt{1-r^2}},r)\,dr\,d\theta = \frac{2}{3}\pi$. (ii) $T(t) = (\cos t, \sin t, 0)$ $\oint_D F\circ T(t)\cdot T'(t)\,dt = \int_0^{2\pi}(\sin t,\cos t,0)\cdot(-\sin t,\cos t,0)\,dt = 0$.

Solution 1.7. $\int_0^1\int_0^{2\pi}\sqrt{(\cos\theta,\sin\theta,0)\times(-r\sin\theta,r\cos\theta,0)}\,d\theta\,dr = \pi$.

Solution 1.8. From Green' theorem

$$\oint_{\partial C} F\circ c(t)\cdot c'(t)\,dt = \iint_C (\nabla\times F)\cdot\mathbf{k}\,dy\,dx.$$

Then $F(x,y) = (2xy-x^2, x+y^2)$ and the first mapping is $T_1(t) = (t,t^2), t\in[0,1]$ so $\int_0^1(2t^3-t^2,t+t^4)\cdot(1,2t)\,dt = \frac{7}{6}$. The second mapping is $T_2(t) = (t,\sqrt{t}), t\in[1,0]$, then $\int_1^0(2t^{\frac{3}{2}}-t^2,2t)\cdot(1,\frac{1}{2\sqrt{t}})\,dt = -\frac{17}{15}$. So the line integral is $\frac{1}{30}$. The double integral is $\iint_C(\nabla\times F)\cdot\mathbf{k}\,dy\,dx = \int_0^1\int_{x^2}^{\sqrt{x}}(0,0,1-2x)\cdot\mathbf{k}\,dydx = \frac{1}{30}$. The double integral is equal to the line integral, the Green' theorem is verified.

Solution 1.9. From Stokes' theorem

$$\oint_{\partial D} F\circ c(t)\cdot c'(t)\,ds = \iint_S (\nabla\times F)(T_\beta)\cdot T_v\times T_u\,dv\,du.$$

Since $\frac{x^2}{2}+\frac{y^2}{2} = 2 \Leftrightarrow x^2+y^2 = 4$, the mapping is $T(t) = (2\cos t, 2\sin t, 2), t\in[2\pi,0]$. Then $-\int_0^{2\pi}(6\sin t,-4\cos t,8\sin t)\cdot(-2\sin t,2\cos t,0)\,dt = \int_0^{2\pi} -12\sin^2 t - 8\cos^2 t\,dt = 20\pi$. With the mapping $T_\beta(r,\theta) = (r\cos\theta, r\sin\theta,\frac{r^2}{2})$, the double integral is $\iint_C(\nabla\times F)(T_\beta)\cdot T_{\beta^r}\times T_{\beta\theta}\,dr\,d\theta = \int_0^2\int_0^{2\pi}(2r\sin\theta+r\cos\theta,0,-\frac{r^2}{2}-3)\cdot(r^2\cos\theta,r^2\sin\theta,-r)\,d\theta dr = 2\pi$. The double integral is equal to the line integral, the Stokes' theorem is verified.

Solution 1.10. From Gauss' theorem

$$\oiint_{\partial W} F\circ T(u,v)\cdot T_v\times T_u\,dv\,du = \iiint_W(\nabla\cdot F)\,dz\,dy\,dx.$$

First we compute $\nabla\cdot F = 2xz^3+2xz^3+4xz^3 = 8xz^3$, then

$$\begin{aligned}\oiint_s F\,ds &= \iiint_B(\nabla\cdot F)dV \\ &= \int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} 8xz^3\,dx\,dy\,dz \\ &= 0.\end{aligned} \tag{1.1}$$

Solutions for Chapter 2

Solution 2.1. $v = -\sigma_2 + \sigma_1 \wedge \sigma_2$, $v = \sigma_2 + 2\sigma_{21}$, $v = 1 + \sigma_1 + 2\sigma_2 - \sigma_{12}$.

Solution 2.2. $a = -\sigma_1 + \sigma_2$, and $b = 2\sigma_1 + 3\sigma_2$.

Solution 2.3. $ab = -\sigma_2 + \sigma_1 = \sigma_1 - \sigma_2$. So $a \wedge b = \sigma_1 - \sigma_2$.

Solution 2.4. $ab = 1 - 3\sigma_{12}$, and $ba = 1 + 3\sigma_{12}$, $a \wedge b = \frac{1}{2}(ab - ba) = -3\sigma_{12}$.

Solution 2.5. (i) $ab = \sigma_2, ba = -\sigma_2$. (ii) $a \cdot b = 0$. (iii) $a \wedge b = \sigma_2$.

Solution 2.6. (i) $a(b+c) = 2$. (ii) $ab = 1 - \sigma_{12}$ and $ac = 1 + \sigma_{12}$, then $ab + ac = 2$. (iii) From (i) and (ii) yes, it is.

Solution 2.7. (i) $a(b+c) = 1 - \sigma_1 + \sigma_2 + 3\sigma_{12}$. $(b+c)a = 1 + \sigma_1 + \sigma 2 + 3\sigma_{12}$. (ii) $a \wedge (b+c) = \sigma_1 - \sigma_2$. (iii) $a \wedge b = -\sigma_2$ and $a \wedge c = 0$, then $a \wedge b + a \wedge c = \sigma_1 - \sigma_2$. (iv) Yes, it is.

Solution 2.8. $a^{-1} = \frac{a}{a \cdot a} = \frac{2 - \sigma_1 + \sigma_1\sigma_2}{2 - 4\sigma_1 + 4\sigma_{12}}$.

Solution 2.9. (i) From the definition, the reversion of a is $a^\dagger = \sigma_2\sigma_1$.

Solution 2.10. Its blades are $< a >_0 = 1$ and $< a >_1 = 0$ $< a >_2 = 2\sigma_{12}$.

Solution 2.11. $a \wedge b = -2\sigma_{12} + \sigma_1$ then $I(a \wedge b) = \sigma_{12}(-2\sigma_2 + \sigma_1) = -2\sigma_1 - \sigma_2$.

Solution 2.12.

$$\begin{aligned} ||a|| &= \sqrt{\lfloor aa^\dagger \rfloor} \\ &= \sqrt{\lfloor (1 + 2\sigma_1 + 3\sigma_2 + 3\sigma_{21})(1 + 2\sigma_1 + 3\sigma_2 - 3\sigma_{21}) \rfloor} \\ &= \sqrt{\lfloor 5 + \cancel{22\sigma_1 - 6\sigma_2} \rfloor} \\ &= \sqrt{5}. \end{aligned} \tag{2.2}$$

Solution 2.13. (i) $a(bc) = -\alpha\sigma_2 - \sigma_1$. (ii) $(ab)c = -\alpha\sigma_2 - \sigma_1$. (iii) From the results (i) and (ii) yes, it is.

Solution 2.14. (i) $Ia = \sigma_1\sigma_2 a = \sigma_2 + \sigma_1$. (ii) $aI = a\sigma_1\sigma_2 = -\sigma_2 - \sigma_1$. (iii) From these results, (i) is a **rotation** of $\frac{\pi}{2}$ in the **clockwise** direction and (ii) is a **rotation** of $\frac{\pi}{2}$ in the **counter-clockwise** direction.

Solution 2.15. (i) $IIa = \sigma_1\sigma_2\sigma_1\sigma_2 a = -2\sigma_1 - 3\sigma_2$. (ii) $aII = a\sigma_1\sigma_2\sigma_1\sigma_2 = -2\sigma_1 - 3\sigma_2$. (iii) From these results, (i) is a **reflection** of π in the **clockwise** direction and (ii) is a **reflection** of π in the **counter-clockwise** direction.

Solution 2.16. $u = \frac{u}{||u||} = \frac{\sigma_1 - 2\sigma_2}{3}$. Then $y = -uxu = \frac{10}{9}\sigma_1 + \frac{5}{9}\sigma_2$.

Solution 2.17. The **line** $L_{x_0}(v)$ is given by

$$L_{x_0=(1,1)}(v) := \{\mathbf{x} \mid (\mathbf{x}-x_0)\wedge v = 0\}$$

$$[(x_1\sigma_1 + x_2\sigma_2) - (\sigma_1+\sigma_2)]\wedge\sigma_1 = 0$$
$$[(x_1-1)\sigma_1 + (x_2-1)\sigma_2]\wedge\sigma_1 = 0$$

The exterior product $(x-x_0)\wedge v = \frac{1}{2}[(x-x_0)v - v(x-x_0)]$,

$$[(x_1-1)\sigma_1 + (x_2-1)\sigma_2]\wedge\sigma_1 = (x_2-1)\sigma_1\sigma_2 = 0 \tag{2.3}$$

From (Eq. 2.3), $x_1 = \mathbb{R}$ and $x_2 = 1$, so the points with the form $(\mathbb{R},1)$ are the solution. Note that the point $(1,1)$ meets the **line** $L_{x_0}(v)$.

Solution 2.18. The **plane** $P_{x_0}(u,v)$ is given by

$$P_{x_0=(2,1)}(u\wedge v) := \{\mathbf{x} \mid (\mathbf{x}-x_0)\wedge(u\wedge v) = 0\} \tag{2.4}$$

$$\begin{aligned}[(x_1\sigma_1 + x_2\sigma_2) - (\sigma_1+2\sigma_2)]\wedge(\sigma_1\sigma_2) = 0\\ [(x_1-1)\sigma_1 + (x_2-2)\sigma_2]\wedge(\sigma_1\sigma_2) = 0\end{aligned} \tag{2.5}$$

From the equation $[(x_1-1)\sigma_1 + (x_2-2)\sigma_2][\sigma_1\sigma_2] - [\sigma_1\sigma_2][(x_1-1)\sigma_1 + (x_2-2)\sigma_2] = 0$. So, the points (x_2-2, x_1-1) are the solution. Note that the point $(2,1)$ meets the **plane** $P_{x_0=(2,1)}(u\wedge v)$.

Solutions for Chapter 3

Solution 3.1. $v = 2\sigma_{123}$, $v = 1 + 3\sigma_{321}$, $v = 1 + \sigma_1 + 2\sigma_2 - \sigma_{12} + \sigma_{123}$.

Solution 3.2. $a = -\sigma_1 + \sigma_2 + \sigma_3$, and $b = 2\sigma_1 + 3\sigma_2 - 3\sigma_3$.

Solution 3.3. $ab = \sigma_{23} - \sigma_{13}$ $ba = \sigma_{23} - \sigma_{23}$. So $a \wedge b = 0$.

Solution 3.4. $ab = -\sigma_{3213} = -\sigma_{21} = \sigma_{12}$, and $ba = -\sigma_{12}$, $a \wedge b = \frac{1}{2}(ab - ba) = -\sigma_{12}$.

Solution 3.5. (i) $ab = \sigma_{21}, ba = -\sigma_{21}$. (ii) $a \cdot b = 0$. (iii) $a \wedge b = \sigma_{21} = -\sigma_{12}$.

Solution 3.6. (i) $a(b+c) = 2\sigma_{31} - \sigma_{32} + \sigma_{12}$. (ii) $ab = \sigma_{31} - \sigma_{32}$ and $ac = \sigma_{31} + \sigma_{12}$, then $ab + ac = 2\sigma_{31} - \sigma_{32} + \sigma_{12}$. (iii) From (i) and (ii) yes, it is.

Solution 3.7. (i) $a(b+c) = 1 - \sigma_1 + \sigma_2 + 3\sigma_{12}$. $(b+c)a = 1 + \sigma_1 + \sigma 2 + 3\sigma_{12}$. (ii) $a \wedge (b+c) = \sigma_1 - \sigma_2$. (iii) $a \wedge b = -\sigma_2$ and $a \wedge c = 0$, then $a \wedge b + a \wedge c = \sigma_1 - \sigma_2$. (iv) Yes, it is.

Solution 3.8. $a^{-1} = \dfrac{a}{a \cdot a} = \dfrac{1 - \sigma_1 + 2\sigma_1\sigma_3}{-2 + 4\sigma_{12} - 4\sigma_3}$.

Solution 3.9. (i) From the definition, the reversion of a is $a^\dagger = -\sigma_1\sigma_3$.

Solution 3.10. Its blades are $< a >_0 = 1$ and $< a >_2 = 2\sigma_{12}$ $< a >_3 = -\sigma_{123}$.

Solution 3.11. $a \wedge b = -2\sigma_3$ then $I(a \wedge b) = \sigma_{123}(-2\sigma_3) = -2\sigma_{12}$.

Solution 3.12. The norm of a is.

$$
\begin{aligned}
||a|| &= \sqrt{\lfloor aa^\dagger \rfloor} \\
&= \sqrt{\lfloor (1 + \sigma_1 + \sigma_2 - \sigma_{21})(1 + \sigma_1 + \sigma_2 + \sigma_{21}) \rfloor} \\
&= \sqrt{\lfloor 4 + \cancel{4\sigma_2} \rfloor} \\
&= \sqrt{2}.
\end{aligned}
\tag{3.6}
$$

Solution 3.13. (i) $a(bc) = -\alpha\sigma_{32} - \sigma_{31}$. (ii) $(ab)c = -\alpha\sigma_{32} - \sigma_{31}$. (iii) From the results (i) and (ii) yes, it is.

Solution 3.14. (i) $Ia = \sigma_1\sigma_2\sigma_3 a = \sigma_{23} + \sigma_{12}$. (ii) $aI = a\sigma_1\sigma_2\sigma_3 = \sigma_{23} + \sigma_{12}$. (iii) From these results, (i) is a **rotation** of $\frac{\pi}{2}$ in the **clockwise** direction and (ii) is a **rotation** of $\frac{\pi}{2}$ in the **counter-clockwise** direction.

Solution 3.15. (i) $IIa = \sigma_{1231232} + \sigma_{1231233} = -\sigma_1 - \sigma_3$. (ii) $aII = \sigma_{2123123} + \sigma_{3123123} = -\sigma_1 - \sigma_3$. (iii) From these results, (i) is a **reflection** of π in the **clockwise** direction and (ii) is a **reflection** of π in the **counter-clockwise** direction.

Solution 3.16. $u = \dfrac{u}{||u||} = \dfrac{\sigma_1 - 2\sigma_2}{3}$. Then $y = -uxu = \dfrac{2}{3}\sigma_1 + \dfrac{8}{9}\sigma_2 + \sigma_3$.

Solution 3.17. The **line** $L_{x_0}(v)$ is given by

$$L_{x0=(0,1,0)}(v) := \{\mathbf{x} \mid (\mathbf{x} - x_0) \wedge v = 0\}$$

$$[(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3) - (0\sigma_1 + \sigma_2 + 0\sigma_3)] \wedge (\sigma_1 + \sigma_2 + \sigma_3) = 0$$
$$[(x_1 - 0)\sigma_1 + (x_2 + 1)\sigma_2 + (x_3 - 0)\sigma_3] \wedge (\sigma_1 + \sigma_2 + \sigma_3) = 0$$

The outer product $(x - x_0) \wedge v = \frac{1}{2}[(x - x_0)v - v(x - x_0)]$,

$$(x_2 - 1)\sigma_{23} + x_1\sigma_{13} = 0 \tag{3.7}$$

From (Eq. 4.11), $x_1 = 0, x_2 = 1,$ and $x_3 = \mathbb{R}$. So, the points with the form $(0,1,\mathbb{R})$ are the solution. Note that the point $(0,1,0)$ meets the **line** $L_{x_0}(v)$.

Solution 3.18. The **plane** $P_{x_0}(u,v)$ is given by

$$P_{x0=(2,1,1)}(u \wedge v) := \{\mathbf{x} \mid (\mathbf{x} - x_0) \wedge (u \wedge v) = 0\} \tag{3.8}$$

$$\begin{aligned} [(x_1\sigma_1 + x_2\sigma_2) - (\sigma_1 + 2\sigma_2)] \wedge (\sigma_1\sigma_2) = 0 \\ [(x_1 - 1)\sigma_1 + (x_2 - 2)\sigma_2] \wedge (\sigma_1\sigma_2) = 0 \end{aligned} \tag{3.9}$$

From (Eq. 4.13), $[(x_1 - 1)\sigma_1 + (x_2 - 2)\sigma_2][\sigma_1\sigma_2] - [\sigma_1\sigma_2][(x_1 - 1)\sigma_1 + (x_2 - 2)\sigma_2] = 0$. So, the points $(x_2 - 2, x_1 - 1)$ are the solution. Note that the point $(2,1,1)$ meets the **plane** $P_{x0=(2,1,1)}(u \wedge v)$.

Solutions for Chapter 4

Solution 4.1. $v = 2,\ v = 1 + 3\sigma_1,\ v = -\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \sigma_4 \wedge \sigma_5 \wedge \sigma_6,\ v = 2\sigma_{21}$, $v = 1 + \sigma_1 + 2\sigma_2 - \sigma_{12} + \sigma_{123456789}$.

Remark 4.1. All of the above vectors are considered to be **multivectors** in their most general sense. So it is avoided to qualify them particularly as **bivectors**, **trivectors** among other adjectives.

Solution 4.2. (i) $a = -\sigma_1 + \sigma_2 + \sigma_3 - \sigma_4 + 2\sigma_5$, and $b = 2\sigma_1 + 3\sigma_2 - 3\sigma_3 + \sigma_4 - \sigma_5 + 3\sigma_6$.

(ii) The line generated by L_{x_0} and the plane generated by P_{x_0}, with the orientation of the vectors v and $u \wedge v$ respectively.

$$L_{x_0=(1,2,3,4)}(v) = \{\mathbf{x} \mid (\mathbf{x} - x_0) \wedge v = (4,3,2,1) = 0\},$$

$$P_{x_0=(1,2,3,4)}(u \wedge v) = \{\mathbf{x} \mid (\mathbf{x} - x_0) \wedge [u = (1,-1,1,-1) \wedge v = (1,2,-2,3)] = 0\}.$$

Solution 4.3. $ab = \sigma_{2345}$, and $ba = \sigma_{2345}$, $a \wedge b = \frac{1}{2}(ab - ba) = 0$.

Solution 4.4. $ab = \sigma_{2345}$, and $ba = \sigma_{2345}$, $a \cdot b = \frac{1}{2}(ab + ba) = \sigma_{2345}$.

Solution 4.5. (i) $ab = \sigma_{12345678}, ba = -\sigma_{56781234} = \sigma_{12345678}$. (ii) $a \cdot b = \sigma_{12345678}$. (iii) $a \wedge b = 0$.

Solution 4.6. (i) $a(b + c) = \sigma_4 - 2\sigma_{234} - \sigma_{134}$. (ii) $ab = -\sigma_{234} - \sigma_{134}$ and $ac = -\sigma_{234} + \sigma_4$, then $ab + ac = \sigma_4 - 2\sigma_{234} - \sigma_{134}$. (iii) From (i) and (ii) yes, it is.

Solution 4.7. (i) $a(b + c) = -\sigma_{234} - \sigma_{34}$. $(b + c)a = \sigma_{234} - \sigma_{34}$. (ii) $a \wedge (b + c) = -\sigma_{234}$. (iii) $a \wedge b = -\sigma_{234}$ and $a \wedge c = -\sigma_{34}$, then $a \wedge b + a \wedge c = -\sigma_{234} - \sigma_{34}$. (iv) Yes, it is.

Solution 4.8. $a^{-1} = \dfrac{a}{a \cdot a} = \dfrac{1 + \sigma_1 + \sigma_1 + \cdots + \sigma_n}{a + \sigma_1 a + \sigma_2 a + \cdots + \sigma_n a} = \dfrac{a}{a + a(\sigma_1 + \sigma_2 + \cdots + \sigma_n}$
$= \dfrac{a}{a + a(a-1)} = \dfrac{a}{a + a^2 - a)} = \dfrac{1}{a}$.

Solution 4.9. (i) From the definition, the reversion of a is $a^\dagger = \sigma_{654321}$.

Remark 4.2. Note that in this case $a^\dagger = -a$.

Solution 4.10. Its blades are $< a >_0 = 1$ and $< a >_2 = 2\sigma_{12}$ $< a >_7 = \sigma_{123456}$.

Solution 4.11. $a \wedge b = -a$ then $I(a \wedge b) = -Ia$.

Solution 4.12. The norm of a is.

$$\begin{aligned} ||a|| &= \sqrt{\lfloor aa^\dagger \rfloor} \\ &= \sqrt{\lfloor (\sigma_1 + \sigma_2 + \cdots + \sigma_n)(\sigma_1 + \sigma_2 + \cdots + \sigma_n) \rfloor} \\ &= \sqrt{\lfloor n + \cancel{\text{vector-residue}} \rfloor} \\ &= \sqrt{n}. \end{aligned} \tag{4.10}$$

Solution 4.13. (i) $a(bc) = \sigma_{12346}$. (ii) $(ab)c = \sigma_{12346}$. (iii) From the results (i) and (ii) yes, it is.

Solution 4.14. (i) $Ia = \sigma_{1234}a =$. (ii) $aI = a\sigma_{1234}$. (iii) From these results, (i) is a **rotation** of $\frac{\pi}{2}$ in the **clockwise** direction and (ii) is a **rotation** of $\frac{\pi}{2}$ in the **counter-clockwise** direction.

Solution 4.15. (i) $IIa = \sigma_{1234}\sigma_{1234}a$. (ii) $aII = a\sigma_{1234}\sigma_{1234}$. (iii) From these results, (i) is a **reflection** of π in the **clockwise** direction and (ii) is a **reflection** of π in the **counter-clockwise** direction.

Solution 4.16. $u = \dfrac{u}{||u||} = \dfrac{\sigma_1 - 2\sigma_2}{3}$. Then $y = -uxu = \dfrac{2}{3}\sigma_1 + \dfrac{8}{9}\sigma_2 + \sigma_3$.

Solution 4.17. The **line** $L_{x_0}(v)$ is given by

$$L_{x_0=(1,1,1,1)}(v) := \{\mathbf{x} \mid (\mathbf{x} - x_0) \wedge v = 0\}$$

$$[(x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 + x_4\sigma_4) - (\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)] \wedge (\sigma_1 + \sigma_4) = 0$$
$$[(x_1 - 1)\sigma_1 + (x_2 + 1)\sigma_2 + (x_3 - 1)\sigma_3 + (x_4 - 1)\sigma_4] \wedge (\sigma_1 + \sigma_4) = 0$$

The exterior product $(x - x_0) \wedge v = \frac{1}{2}[(x - x_0)v - v(x - x_0)]$,

$$[(x_2 - 1)\sigma_{12} + (x_3 - 1)\sigma_{13} - (x_4 - 1)\sigma_{14}] = 0 \tag{4.11}$$

From (Eq. 4.11), $x_1 = \mathbb{R}, x_2 = 1, x_3 = 1,$ and $x_4 = 0$. So, the points with the form $(\mathbb{R}, 1, 1, 1)$ are the solution. Note that the point $(1, 1, 1, 1)$ meets the **line** $L_{x_0}(v)$.

Solution 4.18. The **plane** $P_{x_0}(u, v)$ is given by

$$P_{x_0=(2,1,1,1,1)}(u \wedge v) := \{\mathbf{x} \mid (\mathbf{x} - x_0) \wedge (u \wedge v) = 0\} \tag{4.12}$$

$$\begin{aligned} [(x_1\sigma_1 + x_2\sigma_2) - (\sigma_1 + 2\sigma_2)] \wedge (\sigma_1\sigma_2) = 0 \\ [(x_1 - 1)\sigma_1 + (x_2 - 2)\sigma_2] \wedge (\sigma_1\sigma_2) = 0 \end{aligned} \tag{4.13}$$

From (Eq. 4.13), $[(x_1 - 1)\sigma_1 + (x_2 - 2)\sigma_2][\sigma_1\sigma_2] - [\sigma_1\sigma_2][(x_1 - 1)\sigma_1 + (x_2 - 2)\sigma_2] = 0$. So, the points $(x_2 - 2, x_1 - 1)$ are the solution. Note that the point $(2, 1, 1, 1, 1)$ meets the **plane** $P_{x_0=(2,1,1,1,1)}(u \wedge v)$.

Solutions for Chapter 5

Solution 5.1. The n degree of a w_i form is the term that corresponds to the highest degree in the form. $w_o(x,y,z) = 3+2xyz$ is a $0-$form. $w_i(x,y,z) = 3+2xyz+4dz$ is a $1-$form. $w_2(x,y,z) = 3+2xyz+4dz+dydz$ is a $2-$form. Note this includes terms of a $0-$form and a $1-$form. $w_3(x,y,z) = 2+e^{xyz}dx \wedge dy \wedge dz$. $w_4(x_1,x_2,x_3,x_4) = e^{x_1}dx_1 \wedge d_2 \wedge d_3 \wedge d_4$.

Solution 5.2.

$$\begin{aligned} d(e^{x^2yz}) &= (e^{x^2yz})'_x + (e^{x^2yz})'_y + (e^{x^2yz})'_z \\ &= 2xyze^{x^2yz} + x^2ze^{x^2yz} + x^2ye^{x^2yz} \end{aligned} \tag{5.14}$$

Solution 5.3.

$$\begin{aligned} d(e^{x^2yz}dx + \sin xyzdy) &= d(e^{x^2yz}dx) + d(\sin xyzdy) \\ &= (e^{x^2yz})'_x dx + (e^{x^2yz})'_y dx + (e^{x^2yz})'_z dx \\ &+ (\sin xyz)'_x dx + (\sin xyz)'_y dx + (\sin xyz)'_z dx \\ &= x^2ze^{x^2yz}dydx + x^2ye^{x^2yz}dzdx \\ &+ xz\cos xyzdydx + xy\cos xyzdzdx \end{aligned} \tag{5.15}$$

Solution 5.4.

$$\begin{aligned} d(x^2y+y^3) &= (x^2y+y^3)'_x + (x^2y+y^3)'_y + (x^2y+y^3)'_z \\ &= 2xy + 3y^2 + 0 \end{aligned} \tag{5.16}$$

Solution 5.5.

$$\begin{aligned} d(x^3y+y^3dydz) &= (x^3y+y^3dydz)'_x + (x^3y+y^3dydz)'_y + (x^3y+y^3dydz)'_z \\ &= 3x^2y\,dxdydz \end{aligned} \tag{5.17}$$

Solution 5.6.

$$\begin{aligned} d\left(\frac{-x}{x^2y+y^2}dxdy\right) &= \left(\frac{-x}{x^2y+y^2}dxdy\right)'_x + \left(\frac{-x}{x^2y+y^2}dxdy\right)'_y + \left(\frac{-x}{x^2y+y^2}dxdy\right)'_z \\ &= 0\,dxdydz \end{aligned} \tag{5.18}$$

Solution 5.7. (i)

$$\begin{aligned}
dw &= d(xdx + yzdy + x^3ydz) \\
&= d(xdx) + d(yzdy) + d(x^3ydz) \\
&= d(x)dx + d(yz)dy + d(x^3y)dz \\
&= dxdx + zdydy + ydzdy + 3x^2ydxdz + x^3dydz \\
&= ydzdy + 3x^2ydxdz + x^3dydz \\
&= -ydydz + 3x^2ydxdz + x^3dydz \\
&= 3x^2ydxdz + (x^3 - y)dydz \\
&= 3x^2ydx \wedge dz + (x^3 - y)dy \wedge dz
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
d(xdx) &= (x)'_x dx + (x)'_y dx + (x)'_z dx \\
&= dxdx + 0 + 0 \\
&= dxdx
\end{aligned} \tag{5.20}$$

$$\begin{aligned}
d(yzdy) &= (yz)'_x dy + (yz)'_y dy + (yz)'_z dy \\
&= 0 + zdydy + ydzdy
\end{aligned} \tag{5.21}$$

$$\begin{aligned}
d(x^3ydy) &= (x^3y)'_x dz + (x^3y)'_y dz + (x^3y)'_z dz \\
&= 3x^2ydxdz + x^3dydz + 0
\end{aligned} \tag{5.22}$$

(ii)

$$\begin{aligned}
d(dw) &= d[3x^2ydxdz + (x^3 - y)dydz] \\
&= d(3x^2ydxdz) + d[(x^3 - y)dydz] \\
&= d(3x^2y)dxdz + d(x^3 - y)dydz \\
&= 6xydxdxdz + 3x^2dydxdz + 3x^2dxdydz - dydydz \\
&= 0
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
d(3x^2y)dxdz &= (3x^2y)'_x dxdz + (3x^2y)'_y dxdz + (3x^2y)'_z dxdz \\
&= 6xydxdxdx + 3x^2dydxdz + 0
\end{aligned} \tag{5.24}$$

$$\begin{aligned}
d(x^3 - y)dydz &= (x^3 - y)'_x dydz + (x^3 - y)'_y dydz + (x^3 - y)'_z dydz \\
&= 3x^2dxdydz - dydydz + 0
\end{aligned} \tag{5.25}$$

(iii)

$$\begin{aligned}
(w \wedge \eta) &= (xdx + yzdy + x^3ydz) \wedge (xydz) \\
&= x^2ydxdz + y^2zxdydz + x^3yx^2dzdz \\
&= x^2ydxdz + y^2zxdydz
\end{aligned} \tag{5.26}$$

(iv)

$$\begin{aligned} d(w\wedge\eta) &= d(x^2ydxdz+y^2zxdydz) \\ &= d(x^2ydxdz)+d(y^2zxdydz) \\ &= d(x^2y)dxdz+d(y^2zx)dydz \\ &= -x^2dxdydz+y^2zdxdydz \\ &= (y^2z-x^2)dxdydz \end{aligned} \tag{5.27}$$

$$\begin{aligned} d(x^2y)dxdz &= (x^2y)'_xdxdz+(x^2y)'_ydxdz+(x^2y)'_zdxdz \\ &= 2xydxdxdz+x^2dydxdz+0 \\ &= -x^2dxdydz \end{aligned} \tag{5.28}$$

$$\begin{aligned} d(y^2zx)dydz &= (y^2zx)'_xdydz+(y^2zx)'_ydydz+(y^2zx)'_zdydz \\ &= y^2zdxdydz+2yxzdydydz+y^2xdzdydz \\ &= y^2zdxdydz \end{aligned} \tag{5.29}$$

(v)

From (iv) $d(w\wedge\eta)=dw\wedge\eta+(-1)^kw\wedge d\eta$.

$$\begin{aligned} d(e^{x^2yz})dxdy &= (e^{x^2yz})'_xdxdy+(e^{x^2yz})'_ydxdy+(e^{x^2yz})'_zdxdy \\ &= -x^2ye^{x^2yz}dxdydz \end{aligned} \tag{5.30}$$

Solution 5.8. (i)

$$\begin{aligned} w_{11}\wedge w_{12} &= \frac{1}{2}(w_{11}w_{12}-w_{12}w_{11}) \\ &= (3dx+dy)(e^xdx+2dy) \\ &= 3e^xdx\wedge dx+6dx\wedge dy+e^xdy\wedge dx+2dy\wedge dy \\ &= (6-e^x)dx\wedge dy \\ &= (6-e^x)dxdy \end{aligned} \tag{5.31}$$

(ii)

$$\begin{aligned} d(6-e^x)\wedge dx\wedge dy &= -e^xdx\wedge dx\wedge dy \\ &= 0 \end{aligned} \tag{5.32}$$

Solution 5.9.

$$\begin{aligned} dx\wedge dy &= (-r\sin\theta d\theta+\cos\theta dr)\wedge(r\cos\theta d\theta+\sin\theta dr) \\ &= -r^2\sin\theta\cos\theta d\theta d\theta-r\sin^2\theta d\theta dr+r\cos^2\theta drd\theta+\cos\theta\sin\theta drdr \\ &= (-r\sin^2\theta-r\cos^2\theta)d\theta\wedge dr \\ &= r\,dr\wedge d\theta \end{aligned} \tag{5.33}$$

Solution 5.10. (i)

$$w_4 \wedge w_4 = dx_1 dx_3 \tag{5.34}$$

(ii)

$$\begin{aligned} dw_4 &= d(dx_1 dx_3) \\ &= (dx_1 dx_3)'_{x_1} + (dx_1 dx_3)'_{x_2} + (dx_1 dx_3)'_{x_3} + (dx_1 dx_3)'_{x_4} \\ &= -dx_1 dx_2 dx_3 + dx_1 dx_3 dx_4 \end{aligned} \tag{5.35}$$

Solutions for Chapter 6

Solution 6.1. $\int_D w_0 = \int_1^3 3x^2 + 2x = (17+6) - (1^3+2) = 23 - 3 = 20.$

Solution 6.2. $\int_D w_0 = \int_1^2 \int_1^2 \int_1^2 x^2 + 2xy - z = (2^2+8-2) - (1^3+4-1) = 10$ $-4 = 6.$

Solution 6.3. $\int_D w_1 = \int_0^2 x^4 dx + 3xydy - zdz = \int_0^2 t^4(t)'_t + 3t^3(t^2)'_t - t(t^3)'_t\, dt =$ $\int_0^2 6t^4 + t^4 - 3t^3\, dt = \left[\frac{7}{5}t^4 - \frac{3}{4}t^3\right] = \frac{13}{20}.$

Solution 6.4. $\oint_T F \circ T(t) \cdot T'(t)\, dt = \int_{-\pi}^{\pi} (-t^5, 2\sin t) \cdot (1, 4t^3)\, dt = \int_{-\pi}^{\pi} -t^5 +$ $8t^3 \sin t\, dt = 16\pi(\pi^2 - 6).$

Solution 6.5. $\int_D w_1 = \int_{-\pi}^{\pi} -yxdx + \cos xdy = \int_{-\pi}^{\pi} -t^5(t)'_t + \cos t(t^4)'_t\, dt = \int_{-\pi}^{\pi}$ $-t^5 + 4t^3 \cos t\, dt = 0.$

Solution 6.6. $\int_D w_1 = \int_{-\pi}^{\pi} x^4 + 2x\cos x\, dx = \frac{2}{5}\pi^5.$

Solution 6.7. $\int_D w_2 = \int_0^1 \int_0^{\frac{\pi}{2}} -ydxdy + x^2 dydz\, d\theta dr = \int_0^1 \int_0^{\frac{\pi}{2}} -r\sin\theta \frac{\partial(x,y)}{\partial(r,\theta)}$ $+ r\cos\theta \frac{\partial(y,z)}{\partial(r,\theta)}\, d\theta dr = \int_0^1 \int_0^{\frac{\pi}{2}} -r^2 \sin\theta + r^2 \cos^2\theta \sin\theta\, d\theta dr = -\frac{2}{9}.$

Note 6.1. $\frac{\partial(x,y)}{\partial(r,\theta)} = r$, and $\frac{\partial(y,z)}{\partial(r,\theta)} = \sin\theta.$

Solution 6.8. Using $T(r,\theta) = (r\cos\theta, r\sin\theta, 3)$ with $\theta \in [0, 2\pi]$, $r \in [0, \sqrt{2}]$.

$$\begin{aligned}
\oiint_S F \circ T(r,\theta) \cdot \eta(r,\theta)\, dS &= \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} F(T(r,\theta)) \cdot \frac{\partial T}{\partial r} \times \frac{\partial T}{\partial \theta}\, dr d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} F(T(r,\theta)) \cdot \frac{\partial T}{\partial r} \times \frac{\partial T}{\partial \theta}\, dr d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} (r\cos\theta, r\sin\theta, 1) \cdot (0,0,r) dr d\theta \\
&= 2\pi.
\end{aligned} \tag{6.36}$$

Solution 6.9. $\iint_D w_2 = \int_0^1 \int_0^x x^3 + 2xy\, dy dx = \int_0^1 x^4 + x^3\, dx = \frac{9}{20}.$

Solution 6.10.

$$\int_D w_3 = \int_0^2 \int_0^{2\pi} \int_0^{\pi} [xyzdzdydx]\, dr d\theta d\phi =$$

$$\int_0^1 \int_0^{\pi} \int_0^{\frac{\pi}{2}} r\theta\phi \frac{\partial(z,y,x)}{\partial(r,\theta,\phi)} = \frac{\pi^4}{8}. \tag{6.37}$$

Note 6.2. $\dfrac{\partial(z,y,x)}{\partial(r,\theta,\phi)} = 1$

Solution 6.11. $\iiint_D w_3 = \int_0^1 \int_0^1 \int_0^1 x^4 z + 2xy\,dx\,dy = \dfrac{3}{5}.$

Solution 6.12. $\iiiint_D w_4 = \int_0^{\pi} \int_0^{2\pi} \int_0^{3\pi} \int_0^{4\pi} x_1 x_2 x_3 x_4^2 dx_4\,dx_3\,dx_2\,dx_1 = 24\pi^9.$

Solutions for Chapter 7

Solution 7.1. $\iiint_D w_0 = \int_0^1\int_0^2\int_0^3 xyz\,dz\,dy\,dx = \left(\frac{9}{2}\right)\left(\frac{4}{2}\right)\left(\frac{1}{2}\right) = \frac{9}{2}$. In the Heaviside-Gibbs algebra, this integral represents a volume in the $\mathbb{R}^3$ space or an area in $\mathbb{R}^2$.
In the Geometric algebra this is a $0-$form w_0 integral.

Solution 7.2. $\int_D dw_1 = \int_0^{2\pi} yx\,dx + 2zy\,dy + dz = \int_0^{2\pi} (\cos t\sin t(\cos t)_t' + 2\sin t$ $(\sin t)_t')\,dt = \int_0^{2\pi}(-\sin^2 t\cos t + 2\sin t\cos t)dt$. Now if $F(x,y) = (yx, 2zy, 1)$, $\Rightarrow$ $F\circ C = (\cos t\sin t, 2\sin t, 1)$, then $\int_0^{2\pi}(\cos t\sin t, 2\sin t, 1)\cdot(-\sin t, \cos t, 0)\,dt$. So both integrals are equivalent.

Solution 7.3. $\int_D w_2 = \int_0^1\int_0^{2\pi}[2zdxdy + 3xdydz + 4ydzdx]\,d\theta dr = \int_0^1\int_0^{2\pi}\Big[4 \frac{\partial(x,y)}{\partial(r,\theta)} + r\cos x\frac{\partial(y,z)}{\partial(r,\theta)} + r\sin x\frac{\partial(z,x)}{\partial(r,\theta)}\Big]\,d\theta dr = 24\int_0^1\int_0^{2\pi} r\,d\theta dr = 24\pi.$

Note 7.3. $\frac{\partial(x,y)}{\partial(r,\theta)} = 6r$, $\frac{\partial(y,z)}{\partial(r,\theta)} = 0$, and $\frac{\partial(z,x)}{\partial(r,\theta)} = 0$.

If $F(x,y,z) = (2x, 3y, 4z)$, $T(r,\theta) = (r\cos\theta, r\sin t, 1), r\in[0,1], \theta\in[0,2\pi]$, its Jacobian is 24, then $24\int_0^1\int_0^{2\pi} rd\theta\,dr = 24\pi$. So both integrals are equivalent.

Solution 7.4. $\int_{\partial D} w_1 = \int_0^{2\pi} -4y\,dx + 4x\,dy\;dt = \int_0^{2\pi} -4\sin t(4\cos t)_t' + 4\cos t$
$(4\sin t)_t'\,dt$
$= \int_0^{2\pi} 16\sin^2 t + 16\cos^2 t\,dt = 16\pi.$

$$\begin{aligned} dw_1 &= d(-4y\wedge dx) + d(4x\wedge dy))\\ &= -\left(\frac{\partial 4y}{\partial x}dx + \frac{\partial 4y}{\partial y}dy\right)\wedge dx + \left(\frac{\partial 4x}{\partial x}dx + \frac{\partial 4x}{\partial y}dy\right)\wedge dy\\ &= -\left(\frac{\partial 4y}{\partial x}\right)dx\wedge dx - \left(\frac{\partial 4y}{\partial y}\right)dy\wedge dx\\ &+ \left(\frac{\partial 4x}{\partial x}\right)dx\wedge dy + \left(\frac{\partial 4x}{\partial y}\right)dy\wedge dy\\ &= 8dxdy \end{aligned} \tag{7.38}$$

Now, we parameterize $c(r,\theta) = (r\cos\theta, r\sin\theta, 1)$
$\int_D dw_1 = \int_0^1\int_0^{2\pi}[8dxdy]\,d\theta dr = \int_0^1\int_0^{2\pi}\left[8\frac{\partial(x,y)}{\partial(r,\theta)}\right]d\theta dr = \int_0^1\int_0^{2\pi} 8r\,d\theta dr$
$= 16\pi.$

Note 7.4. $\dfrac{\partial(x,y)}{\partial(r,\theta)} = r.$

Green's theorem is verified.

Solution 7.5.

$$\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dydx = \int_0^2 \int_0^{2-x} 2xy^2 - xdy\, dx$$
$$= -\frac{4}{15}$$

$$\begin{aligned} \int_{\partial D} P \circ c(t)\, \frac{x(t)}{dt} + Q \circ c(t)\, \frac{y(t)}{dt} &= \int_{\partial D_1} P \cdot c_1(t) \frac{dx}{dt} + Q \cdot c_1(t) \frac{dy}{dt} \\ &+ \int_{\partial D_2} P \cdot c_2(t) \frac{dx}{dt} + Q \cdot c_2(t) \frac{dy}{dt} \\ &+ \int_{\partial D_3} P \cdot c_3(t) \frac{dx}{dt} + Q \cdot c_3(t) \frac{dy}{dt} \\ &= 0 - \frac{4}{15} + 0 \\ &= -\frac{4}{15} \end{aligned} \tag{7.39}$$

Where $c_1(t) = (t,0), t \in [0,2]$, $c_2(t) = (2-t,t), t \in [0,2]$, and $c_3(t) = (0,t), t \in [2,0]$.

$$\begin{aligned} \int_{\partial D_1} P \cdot c_1(t) \frac{dx}{dt} + Q \cdot c_1(t) \frac{dy}{dt} &= \int_0^2 xy(0) + x^2y^2(2)dt \\ &= \int_0^2 0(1) + 0(0)dt \\ &= 0 \end{aligned}$$

$$\begin{aligned} \int_{\partial D_2} P \cdot c_2(t) \frac{dx}{dt} + Q \cdot c_2(t) \frac{dy}{dt} &= \int_0^2 xy(-1) + x^2y^2(1)dt \\ &= \int_0^2 -t(1-t) + (1-t)^2(t^2)dt \\ &= -\frac{4}{15} \end{aligned} \tag{7.40}$$

$$\begin{aligned} \int_{\partial D_3} P \cdot c_3(t) \frac{dx}{dt} + Q \cdot c_3(t) \frac{dy}{dt} &= \int_2^0 xy(1) + x^2y^2(0)dt \\ &= \int_2^0 0(0) + 0(1)dt \\ &= 0 \end{aligned}$$

Solution 7.6. $\int_{\partial D} w_1 = \int_0^{2\pi} y\,dx + e^z\,dy + x\,dz\,dt = \int_0^{2\pi} \sin t(\cos t)'_t + e^1$
$(\sin t)'_t + \cos t(1)'_t\,dt = \int_0^{2\pi} -\sin^2 t + e\cos t\,dt = -\pi.$

$$\begin{aligned}
dw_1 &= d(y \wedge dx) + d(e^z \wedge dy) + d(x \wedge dz) \\
&= \left(\frac{\partial y}{\partial x}dx + \frac{\partial y}{\partial y}dy + \frac{\partial y}{\partial z}dz\right) \wedge dx \\
&+ \left(\frac{\partial e^z}{\partial x}dx + \frac{\partial e^z}{\partial y}dy + \frac{\partial e^z}{\partial z}dz\right) \wedge dy \\
&+ \left(\frac{\partial x}{\partial x}dx + \frac{\partial x}{\partial y}dy + \frac{\partial x}{\partial z}dz\right) \wedge dz \\
&= \left(\frac{\partial y}{\partial x}\right) dx \wedge dx + \left(\frac{\partial y}{\partial y}\right) dy \wedge dx + \left(\frac{\partial y}{\partial z}\right) dz \wedge dx \\
&+ \left(\frac{\partial e^z}{\partial x}\right) dx \wedge dy + \left(\frac{\partial e^z}{\partial y}\right) dy \wedge dy + \left(\frac{\partial e^z}{\partial z}\right) dz \wedge dy \\
&= \left(\frac{\partial x}{\partial x}\right) dx \wedge dz + \left(\frac{\partial x}{\partial y}\right) dy \wedge dz + \left(\frac{\partial x}{\partial z}\right) dz \wedge dz \\
&= -dxdy + dzdx + e^z dzdy
\end{aligned} \tag{7.41}$$

Now, we parameterize $c(r,\theta) = (r\cos\theta, r\sin\theta, 1)$
$\int_D dw_1 = \int_0^1 \int_0^{2\pi} [-dxdy + dzdx + e^z dzdy]\,d\theta dr = \int_0^1 \int_0^{2\pi} \Big[-\frac{\partial(x,y)}{\partial(r,\theta)}$
$+\frac{\partial(z,x)}{\partial(r,\theta)} + e^1 \frac{\partial(z,y)}{\partial(r,\theta)}\Big] d\theta dr = \int_0^1 \int_0^{2\pi} -r\,d\theta dr = -\pi.$

Note 7.5. $\frac{\partial(x,y)}{\partial(r,\theta)} = r$, $\frac{\partial(z,x)}{\partial(r,\theta)} = 0$, and $\frac{\partial(z,y)}{\partial(r,\theta)} = 0$.

Stokes's theorem is verified.

Solution 7.7. If $T(\theta,r) = (r\cos t, r\sin t, 1 - r\cos t - r\sin t)$,

$$\frac{\partial(T_y,T_z)}{\partial(\theta,r)} = \begin{vmatrix} r\cos\theta & \sin\theta \\ r\sin\theta & -r\cos\theta \end{vmatrix} \tag{7.42}$$

$$\frac{\partial(T_z,T_x)}{\partial(\theta,r)} = \begin{vmatrix} r\sin\theta & -r\cos\theta \\ -r\sin\theta & \sin\theta \end{vmatrix} \tag{7.43}$$

$$\frac{\partial(T_x,T_y)}{\partial(\theta,r)} = \begin{vmatrix} -r\sin\theta & \sin\theta \\ r\cos\theta & \sin\theta \end{vmatrix} \tag{7.44}$$

$$\begin{aligned}
\iint_S dw &= \int_0^1 \int_0^{2\pi} (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\, dydz + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\, dzdx + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\, dxdy \\
&= \int_0^1 \int_0^{2\pi} (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}) \circ T(\theta, r)\, \frac{\partial (T_y, T_z)}{\partial (\theta, r)} \\
&+ (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}) \circ T(\theta, r)\, \frac{\partial (T_z, T_x)}{\partial (\theta, r)} \\
&+ (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \circ T(\theta, r)\, \frac{\partial (T_x, T_y)}{\partial (\theta, r)} \\
&= \int_0^1 \int_0^{2\pi} (0) \frac{\partial (T_y, T_z)}{\partial (\theta, r)} + (0) \frac{\partial (T_z, T_x)}{\partial (\theta, r)} + (0) \frac{\partial (T_x, T_y)}{\partial (\theta, r)}\, d\theta dr \\
&= 0
\end{aligned} \tag{7.45}$$

$$\begin{aligned}
\int_{\partial D} P \cdot c(t) \frac{dx}{dt} + Q \cdot c(t) \frac{dy}{dt} + R \cdot c(t) \frac{dy}{dt} &= \int_0^{2\pi} (\cos t)(-\sin t) \\
&+ (\sin t)(\cos t) \\
&+ (1 - \cos t - \sin t)(\sin t - \cos t) dt \\
&= 0
\end{aligned} \tag{7.46}$$

Solution 7.8. $\int_D w_2 = \int_0^{\pi} \int_0^{2\pi} [xz\,dxdy - xydxdz - dydz]\, d\theta d\phi = \int_0^{\pi} \int_0^{2\pi} \Big[$ $\cos\theta \sin\phi \cos\phi \frac{\partial (y,x)}{\partial (r,\theta)} + \cos\theta \sin\phi \sin\theta \sin\phi \frac{\partial (z,x)}{\partial (r,\theta)} + \frac{\partial (z,y)}{\partial (r,\theta)} \Big] d\theta d\phi$ $= \int_0^{\pi} \int_0^{2\pi} d\theta d\phi = -\cos\theta \sin\phi \cos\phi \sin\phi \cos\phi - \cos\theta \sin\phi \sin\theta \sin\phi \sin^2\phi \sin\theta$ $-\sin^2\phi \cos\theta\ d\theta\, d\phi = -\int_0^{\pi} \int_0^{2\pi} \cos\theta \cos^2\phi \sin^2\phi + \cos\theta \sin^3\phi \sin^2\theta \cos\theta$ $+\sin^2\phi\ d\theta\, d\phi = 0.$

Note 7.6. $\frac{\partial (x,y)}{\partial (\theta,\phi)} = -\sin\phi \cos\phi$, $\frac{\partial (z,x)}{\partial (\theta,\phi)} = -\sin^2\phi \sin\theta$, and $\frac{\partial (z,y)}{\partial (\theta,\phi)} = \sin^2\phi$ $\cos\theta$.

$$
\begin{aligned}
dw_2 &= -d(dydz) + d(xydzdx) + d(xzdxdy)\\
&= (-\frac{\partial 1}{\partial x}dx - \frac{\partial 1}{\partial y}dy - \frac{\partial 1}{\partial z}dz) \wedge (dy \wedge dz)\\
&+ (\frac{\partial xy}{\partial x}dx + \frac{\partial xy}{\partial y}dy + \frac{\partial xy}{\partial z}dz) \wedge (dz \wedge dx)\\
&+ (\frac{\partial xz}{\partial x}dx + \frac{\partial xz}{\partial y}dy + \frac{\partial xz}{\partial z}dz) \wedge (dx \wedge dy)\\
&= -\frac{\partial 1}{\partial x}dx(dy \wedge dz) - \frac{\partial 1}{\partial y}dy(dy \wedge dz) - \frac{\partial 1}{\partial z}dz(dy \wedge dz)\\
&+ \frac{\partial xy}{\partial x}dx(dz \wedge dx) + \frac{\partial xy}{\partial y}dy(dz \wedge dx) + \frac{\partial xy}{\partial z}dz(dz \wedge dx) \qquad (7.47)\\
&+ \frac{\partial xz}{\partial x}dx(dx \wedge dy) + \frac{\partial xz}{\partial y}dy(dx \wedge dy) + \frac{\partial xz}{\partial z}dz(dx \wedge dy)\\
&= -\frac{\partial 1}{\partial x}dxdydz - \frac{\partial 1}{\partial y}dydydz - \frac{\partial 1}{\partial z}dzdydz\\
&+ \frac{\partial xy}{\partial x}dxdzdx + \frac{\partial xy}{\partial y}dydzdx + \frac{\partial xy}{\partial z}dzdzdx\\
&+ \frac{\partial xz}{\partial x}dxdxdy + \frac{\partial xz}{\partial y}dydxdy + \frac{\partial xz}{\partial z}dzdxdy\\
&= 2x\,dxdydz
\end{aligned}
$$

Ahora aplicamos la parametrizacion $T(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$, $\theta \in [0,2\pi]$, $\phi \in [0,\pi]$, and $\rho \in [0,1]$.

$\int_D dw_2 = \int_0^1 \int_0^\pi \int_0^{2\pi} 2xdxdydz\,d\theta\,d\phi\,d\rho = \int_0^1 \int_0^\pi \int_0^{2\pi} 2\rho\cos\theta\sin\phi$
$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \int_0^1 \int_0^\pi \int_0^{2\pi} \rho\cos\phi 2\rho\cos\theta\sin\phi d\theta\,d\phi\,d\rho = 2\rho^2\cos\phi\sin\phi$
$d\phi\,d\rho = 0.$

Note 7.7. $\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho\cos\phi.$

Gauss's theorem is verified.

Solution 7.9.

$$
\begin{aligned}
\int_\Omega &= \iiint_\Omega (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})\,dz\,dy\,dx\\
&= \int_{-1}^{-1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 8xyz\,dz\,dy\,dx \qquad (7.48)\\
&= 0
\end{aligned}
$$

$$\begin{aligned}\iint_{\partial\Omega} d\Omega &= \int_0^{\pi}\int_0^{2\pi} P\circ T\,\frac{\partial(T_y,T_z)}{\partial(\theta,\phi)} + Q\circ T\,\frac{\partial(T_z,T_x)}{\partial(\theta,\phi)} + R\circ T\,\frac{\partial(T_x,T_y)}{\partial(\theta,\phi)}\,d\theta\,d\phi \\ &= \int_0^{\pi}\int_0^{2\pi} \cos^2\theta\sin^2\phi\,\frac{\partial(T_y,T_z)}{\partial(\theta,\phi)} + \sin^2\theta\sin^2\phi\,\frac{\partial(T_z,T_x)}{\partial(\theta,\phi)} + \cos^2\phi \\ &\frac{\partial(T_x,T_y)}{\partial(\theta,\phi)}d\theta d\phi \\ &= \int_0^{\pi}\int_0^{2\pi} (\cos^2\theta\sin^2\phi)(-\cos\theta\sin^2\phi) + (\sin^2\theta\sin^2\phi)(-\sin^2\phi) \\ &+ (\cos^2\phi)(-\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi)d\theta d\phi \\ &= 0\end{aligned} \tag{7.49}$$

Note 7.8. The sign depends on the orientation.

$$\begin{aligned}\frac{\partial(T_y,T_z)}{\partial(\theta,r)} &= \begin{vmatrix}\cos\theta\cos\phi & \sin\theta\cos\phi \\ 0 & -\sin\phi\end{vmatrix} \\ &= -\cos\theta\sin^2\phi\end{aligned} \tag{7.50}$$

$$\begin{aligned}\frac{\partial(T_z,T_x)}{\partial(\theta,r)} &= \begin{vmatrix}0 & -\sin\phi \\ -\sin\theta\sin\phi & \cos\theta\cos\phi\end{vmatrix} \\ &= -\sin^2\phi\end{aligned} \tag{7.51}$$

$$\begin{aligned}\frac{\partial(T_x,T_y)}{\partial(\theta,r)} &= \begin{vmatrix}-\sin\theta\sin\phi & \cos\theta\cos\phi \\ \cos\theta\sin\phi & \sin\theta\cos\phi\end{vmatrix} \\ &= -\sin^2\theta\sin\phi\cos\phi - \cos^2\theta\sin\phi\cos\phi\end{aligned} \tag{7.52}$$

Solution 7.10.

$$\begin{aligned}\int_D w_3 &= \int_0^1\int_0^2\int_0^3 [x_3x_4\,dx_1\,dx_2\,dx_3]\,du_1du_2du_3 \\ &= \int_0^1\int_0^2\int_0^3 \left[u_1u_3\frac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)}\right]du_1du_2du_3 \\ &= \int_0^1\int_0^2\int_0^3 u_1u_3du_1du_2du_3 \\ &= \frac{9}{2}\end{aligned} \tag{7.53}$$

Note 7.9. $\dfrac{\partial(x_1,x_2,x_3)}{\partial(u_1,u_2,u_3)} = 1.$

$$
\begin{aligned}
dw_3 &= d(x_1x_4\, dx_1dx_2dx_3 + x_2x_3\, dx_3dx_4dx_1) \\
&= (\frac{\partial x_1x_4}{\partial x_1}dx_1 + \frac{\partial x_1x_4}{\partial x_2}dx_2 + \frac{\partial x_1x_4}{\partial x_3}dx_3 + \frac{\partial x_1x_4}{\partial x_4}dx_4) \wedge (dx_1 \wedge dx_2 \wedge dx_3) \\
&+ (\frac{\partial x_2x_3}{\partial x_1}dx_1 + \frac{\partial x_2x_3}{\partial x_2}dx_2 + \frac{\partial x_2x_3}{\partial x_3}dx_3 + \frac{\partial x_2x_3}{\partial x_4}dx_4) \wedge (dx_3 \wedge dx_4 \wedge dx_1) \\
&= \frac{\partial x_1x_4}{\partial x_1}dx_1(dx_1 \wedge dx_2 \wedge dx_3) + \frac{\partial x_1x_4}{\partial x_2}dx_2(dx_1 \wedge dx_2 \wedge dx_3) \\
&+ \frac{\partial x_1x_4}{\partial x_3}dx_3(dx_1 \wedge dx_2 \wedge dx_3) + \frac{\partial x_1x_4}{\partial x_4}dx_4(dx_1 \wedge dx_2 \wedge dx_3) \\
&= \frac{\partial x_2x_3}{\partial x_1}dx_1(dx_3 \wedge dx_4 \wedge dx_1) + \frac{\partial x_2x_3}{\partial x_2}dx_2(dx_3 \wedge dx_4 \wedge dx_1) \\
&+ \frac{\partial x_2x_3}{\partial x_3}dx_3(dx_3 \wedge dx_4 \wedge dx_1) + \frac{\partial x_2x_3}{\partial x_4}dx_4(dx_3 \wedge dx_4 \wedge dx_1) \\
&= \frac{\partial x_1x_4}{\partial x_1}dx_1dx_1dx_2dx_3 + \frac{\partial x_1x_4}{\partial x_2}dx_2dx_1dx_2dx_3 \\
&+ \frac{\partial x_1x_4}{\partial x_3}dx_3dx_1dx_2dx_3 + \frac{\partial x_1x_4}{\partial x_4}dx_4dx_1dx_2dx_3 \\
&+ \frac{\partial x_2x_3}{\partial x_1}dx_1dx_3dx_4dx_1 + \frac{\partial x_2x_3}{\partial x_2}dx_2dx_3dx_4dx_1 \\
&+ \frac{\partial x_2x_3}{\partial x_3}dx_3dx_3dx_4dx_1 + \frac{\partial x_2x_3}{\partial x_4}dx_4dx_3dx_4dx_1 \\
&= -(x_1 + x_3)\, dx_1dx_2dx_3dx_4
\end{aligned}
\tag{7.54}
$$

Now, we parameterize $T(u_1, u_2, u_3, u_4) = (u_1, u_2, u_3, u_4)$,$u_1 \in [0, \alpha_1]$, $u_2 \in [0, \alpha_2]$, $u_3 \in [0, \alpha_3]$, and $u_4 \in [0, \alpha_4]$.
$\int_D dw_3 = \int_0^{\alpha_1} \int_0^{\alpha_2} \int_0^{\alpha_3} \int_0^{\alpha_4} (x_1 + x_3)\, dx_1dx_2dx_3dx_4 = \frac{9}{2}$. Where $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$, and $\alpha_4 = -1 - \frac{-\sqrt{7}}{2}$, or $\alpha_4 = \frac{1}{2}\left(\sqrt{7} - 2\right)$.

The Fundamental Theorem of Calculus is verified.

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