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NUMBERAMA

RECREATIONAL NUMBER THEORY IN THE SCHOOL SYSTEM

Elliot Benjamin

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Numberama
Recreational Number Theory In The School System

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BENTHAM SCIENCE

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This book is dedicated to my son, Jeremy.

FOREWORD

With all the push toward applications of mathematics where some are at best artificial, it is refreshing to find a text that does not have the pretense of giving any real applications, but rather a book on number theory just for fun. The conception of the book, *Numberama*, could have been conceived by the first real number theorist, P. Fermat, through a bunch of problems (without any thought of applications).

This book is a text about problems in number theory intended to aid teachers from early elementary school to early high school in giving an appreciation of number theory to their students. The text is divided into four parts and an Appendix: Chapter 1 is devoted to several basic problems in number theory which can be appreciated using only elementary arithmetic: addition, subtraction, multiplication, and division. Chapters 2, 3, and 4 are devoted to board games based on the problems in Chapter 1, where each chapter requires successively higher level arithmetic skills to play the games in the chapter. For each of the problems given in Chapter 1, there is a code indicating the level of skills needed to work on at least parts of the problems. Teachers are given plenty of advice as to how to present the material to the students. The Appendix includes excerpts from various student and teacher participants in Dr. Benjamin's Numberama program, which describes both the benefits and the joy they received from participating in this program.

A couple of the problems included are the following, (some going back to antiquity): Finding perfect numbers. Here the students get to use their skills at multiplication and division, as well as being exposed to prime numbers. This problem leads, of course, to open questions such as the existence of infinitely many even perfect numbers and also the existence of at least one odd perfect number. There is also the problem of representing integers as the sum of two squares, a problem which Fermat himself worked on. Again the teachers are given hints as to how to proceed.

Chapters 2, 3, and 4 are devoted to 19 different board games concerning the problems in Chapter 1, which seek to hone the skills of the students, involving many of the properties of numbers given in the first chapter. The text is written well and seems to be accurate. I would certainly recommend it to teachers interested in enriching the mathematics content and honing basic arithmetic skills of the students.

Chip Snyder, Ph.D.
Professor of Mathematics,
University of Maine,
Orono,
Maine

PREFACE

It has been nearly 30 years since I began working on my *Numberama* book. However, in spite of the enormous developments in technology and social media over the past 30 years, the essential theme of my book remains intact. The essential theme is that mathematics can be a stimulating, challenging, and thoroughly enjoyable recreational mental activity to enhance and enrich substantial and creative thinking for children in our school system. As I describe in the Appendix, I have experienced a wide variety of appreciative and enthusiastic responses to my Numberama program over the years, ranging across teacher workshops, teacher education programs, children in gifted programs, children in regular classes, liberal arts college instruction, and family math workshops. I have also effectively utilized my Numberama program at a mental hospital for children, a senior citizen center, as a supplement in my algebra classes, and as an example of creative thinking in my psychology classes. Most recently, I found myself giving a "perfect number lesson" on a napkin at a restaurant at a spiritual development workshop I attended. I had the workshop presenter and some of the participants and staff enraptured, and I realized that Numberama is deeply ingrained in me, wherever I go and whatever I do. I am still a pure mathematician, and I practice what I preach. Doing mathematics for me is refreshing, stimulating, meditative, and enjoyable. I occasionally make use of technology to try to find examples of some of my pure mathematics algebraic number theory results, but this is always very secondary, as the priority is on my "thinking." And this is the exact same philosophy I promote in my Numberama program in regard to the use of technology. Technology is a wonderful tool, but it is essential for the human being to be in control of the technology and not the other way around. Thus finding interesting, surprising, and enticing patterns of numbers, with the assistance of arithmetic calculators when the numbers invariable get very large, is a natural part of my Numberama problems. But what is most important here is the discovery of the patterns themselves, using technology to enhance the discovery.

With this in mind, I am excited to now be offering my *Numberama* book as an e-book through Bentham publications. I welcome feedback from anyone who is using my Numberama problems and games, and I hope that I have succeeded in transmitting the joys of searching for captivating patterns of numbers in my book.

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Minneapolis,
USA

ACKNOWLEDGEMENTS

I would like to take this opportunity to give special thanks to a few individuals who truly have made this book possible. First, I am greatly indebted to Dr. Chip Snyder—my mathematical mentor.

I have been working with Dr. Snyder in the pure mathematics discipline of Algebraic Number Theory since I moved to Maine in 1985, having earned my Ph.D. in this mathematical discipline in 1996. We have worked together all these years for essentially one reason—we both enjoyed it—and still do. I learned first-hand the joyful, challenging, frustrating, and transcendental experience of what it means to be a research mathematician. Thus I have been able to practice what I preach.

Next, I must give my heartfelt thanks to my son, Jeremy. As I describe in the Introduction to Chapter 2, it was Jeremy who inspired the games of recreational number theory. It was also my son Jeremy who lived through many of the recreational number theory problems—from ages 7 through 11. He has been wonderfully responsive and patient with his rather unusual father, and I love him dearly.

I must also express my appreciation to a student in my first teacher education in mathematics class at the University of Maine in 1990—Ethel Hill. Ethel thoroughly enjoyed my "special problems." Ethel has also played an enormously important role in making this book a reality—she has done all of the typing! She learned how to put everything on the computer—including the games—in her spare time while working as administrative assistant to the Dean of the College of Education at the University of Maine. She has done this with a marvelous spirit and has encouraged me to persevere in making this book into a vehicle to help in the transformation of mathematics from drudgery to fun. I hope she continues to involve herself in the next stage of marketing this book.

The next individual I give my thanks to is no longer with us—her name is Stephanie Pall. My friend Stephanie was the person who inspired me to find a way to convey to others the joys I have experienced from mathematics. She enabled me to look deeply inside myself, to be who I truly am in my career of teaching mathematics. This occurred in 1988 as I began playing with many of the problems in David Wells' (1986) book.

The Penguin Dictionary of Curious and Interesting Numbers, for the purpose of improving my mathematics teaching at Unity College. I subsequently found a number of additional resources that were helpful to me in formulating both my number theory problems and related teaching methods (Adams & Goldstein, 1976; Beiler, 1966; Brown, 1973, 1976, 1983; Dence, 1983; Dewey, 1933; King, 1993; Miller & Heeren, 1961; National Council of Mathematics, 1981, 1984, 1991; Neill, 1960; Postman & U Weingartner, 1969; Rogers, 1969; Walter & Brown, 1971; Walter & Brown, 1977).

Stephanie's untimely death prevented her from seeing where her faith in me has led. But I give tribute to her now; I will forever be indebted to her for the gift she has given me.

I wish to thank all the students in my Finite Math classes at Unity College, my mathematics teacher education classes at the University of Maine, and the teachers and children who participated in my "Mathworks" program in Belfast, Maine, during the 1991−1992 school year.

I also would like to thank my Swanville, Maine, friends who so patiently indulged me in my mathematical entertainments on many a lazy Sunday afternoon in their home—Steve and Kate Webster.

Finally, I give my thanks to Thomas Hathaway Nason from the Word Shop in Orono, Maine, who patiently and effectively put finishing computer touches on the book, a task which turned out to be far more demanding than originally anticipated, and to Kay Retzlaff, my present book consultant who is responsible for the book's new layout and design.

There are many more individuals I am not mentioning who have helped me to form my ideas about both mathematics and mathematics education. I give a final note of thanks to those many unnamed individuals.

Description of Skill Levels

The following letters will be used to denote the designated math skill levels. All problems and games are followed by the appropriate skill level; problems followed by two or more skill levels imply that they can be used in various degrees of skill complexity. It should be noted that all students can gain value from working on problems from previous skill levels:

1. addition of two-digit numbers
2. general addition and subtraction
3. one-digit multiplication
4. general multiplication
5. multiplication division by 2
6. multiplication division by one-digit number
7. multiplication division in general
8. fractions
9. signed numbers
10. algebra

INTRODUCTION TO THE BOOK

It is now over 20 years since I wrote the above acknowledgments for this book, as well as the basis of this introduction. However, it is a tribute to the timeless nature of these Recreational Number Theory problems and games that I have designated with the title of *Numberama*, that there is little I feel the need of adding to at this time. My philosophy of "math for fun" has not changed, and I am still collaborating with my ex-Ph.D mathematics mentor Dr. Chip Snyder as we continue to work together, publishing papers in the field of algebraic number theory. I have utilized my Numberama problems and games in diverse educational settings, inclusive of various elementary school classrooms, gifted and talented school programs, developmental mathematics classes at colleges and universities, teacher workshops, and even at a senior retirement home. The past few years I have utilized the Subsets & Circles problem (see Problem #1 in Chapter 1) in my Introductory Psychology classes to illustrate the experience of creative thinking. The results of all my Numberama explorations with both students and teachers have been overwhelmingly positive, and I have received many written descriptions of the benefits that participants have received from their experiences in my Numberama program, a sample of which I have included in the Appendix.

I believe that today, more than ever, it is so very important to not let our children lose (or never experience) the intrinsic joy of doing mathematics. Our technology is so sophisticated that it is all too easy for both our children and ourselves to discontinue our "thinking" and let our computer gadgets "think" for us. But there is an inherent potential joy in thinking, and I am thankful that I continue to experience this inherent joy of mathematical thinking in my pure mathematics field of algebraic number theory. And it continues to be part of my mission in life to convey the inherent joy of mathematical thinking to children in the context of Numberama Recreational Number Theory problems and games in the school system, and to people of all ages, through my *Numberama* book.

As a child I enjoyed adding numbers in my head. People were amazed at how quickly I could do so without using pencil or paper. Throughout school I enjoyed math and, as a result, I was good at it. It was no surprise to anyone when I decided to become a math teacher; however, I soon realized that the intrinsic rewards I received from studying mathematics were by no means a common experience for other students. After teaching elementary and high school, college, and in various adult education programs, I came to the conclusion that the vast majority of our population has a very limited perspective of what mathematics is truly all about.

Mathematics can certainly be an extremely pragmatic science, chock full of useful applications in virtually every field studied; however, there is another side to mathematics. Pure mathematics can be described as an art form, in the same way music, art, dance and theater are arts. Nearly every professor of mathematics knows this deep down in his/her heart. Mathematics is truth and beauty within the spirit of the mind. The natural process of thinking is inherently pleasurable. Pressures, grades, competition, *etc*., can destroy this potential intrinsic pleasure. What I refer to as a "natural dimension of mathematics" is doing math for the pure enjoyment of learning and discovering. Math can be fun.

This book attempts to impart the enjoyment of mathematics to the children in our schools, whether these schools are at home or part of a public or private system. The branch of mathematics that literally plays with numbers is known as number theory. Topics in number theory range from the highly theoretical, employing deep layers of abstract mathematical proof, to questions about numbers that any school child learning arithmetic can understand. These questions are enticing, adventuresome, challenging, and most important of all—fun.

I call this form of number theory, recreational number theory. Most of the problems described in Chapter 1 in this book can be worked by children who know how to add, subtract, multiply, and divide. A number of the problems do not even require division; some of the problems only require addition. There are also problems for children first learning fractions, and in many of the problems I have given suggestions on how they can be formulated into algebra problems for older students, in junior and senior high school. For each problem, the exact prerequisite skills are indicated. The general format is described at the end of the Table of Contents. The problems I have chosen to describe are by no means exhaustive. An examination of the bibliography I have included will give the interested reader some supplementary material. There is a place in our schools for "math for fun" problems. The earlier such problems are introduced, the easier it will be for a child to learn the basics of arithmetic. Working on these problems requires a lot of practice in nearly all of the arithmetic skills that are now being taught in the elementary schools. But the practice and drill are made fun through the discovery of patterns, formulas, unusual numbers, *etc*. The approach I am recommending is very much like playing a game.

Chapters 2, 3, and 4 consist of a series of 19 games based upon the ideas from recreational number theory introduced in Chapter 1, with each chapter requiring successively higher arithmetic skills for children to play the games included in the chapter. These games hones the skills of the students, involving many of the properties of numbers given in the first chapter. Once again, the games are by no means exhaustive, but merely serve as a rough.

Mathematics can certainly be an extremely pragmatic science, chock full of useful applications in virtually every model of how many math ideas can be made into games where children are joyfully practicing their arithmetic skills while playing the game. The prerequisite skills necessary to play the games are listed for each game, in the same format described for the problems in Chapter 1. These games serve to reinforce ideas encountered in the problems. Although a major emphasis of recreational number theory is the elementary school, this is by no means the only place where it can be used. I have purposely included many generalizations to algebraic formulas in order to make the point that recreational number theory can be used throughout the school years.

Junior high and high school students can be taught to use their newly acquired algebra skills to generate their own algebraic formulas that describe experimental facts about numbers that they have gathered together. This approach to teaching algebra is a radical change from the often tedious, monotonous, and overly pragmatic way that algebra is generally taught in our school system. I am by no means recommending that all of the traditional material in arithmetic or algebra be deleted from our schools; rather I am advocating an exciting new tool and method of education that can be used to help our children learn many of these skills. The key word is "balance." There is a place for lecture, a place for tradition, and also a place for process, adventure, and discovery.

Another challenge is to successfully use the discovery approach of recreational number theory with college students in the area known as developmental mathematics, which is little more than arithmetic and algebra for college students and adults going back to school. Community colleges and continuing education departments are teaching more of arithmetic and algebra to their students than any other kind of math. For much of my career as a mathematics professor, this was my own specialized field, and math anxiety, resistance, built-up failures, *etc*., are painfully high in this student population. To enable these students to view mathematics as a pleasurable pastime is indeed challenging.

This is the challenge this book is intended to meet. I have seen extremely dramatic results with students who hated math all of their lives. The prospect that they could now play with numbers for the purpose of making joyful discoveries was a welcome change of pace for them; however, the results were best when I was able to use the discovery approach exclusively without having to worry about required topics, exams, and follow-up courses. I realize this is not the typical situation our students are in, and throughout my mathematics college teaching years.

I continued to search for an effective way of balancing the old and the new; *i.e.*, to incorporate the ideas and processes of recreational number theory within the traditional format of our developmental mathematics courses.

I hope that you will find value in the following problems and games, whether you are a math teacher, prospective math teacher, math student, parent, or interested reader in general. I welcome any feedback you have, and look forward to hearing from you.

INTRODUCTION TO THE GAMES

When my son Jeremy was 7 years old, he made me a little math game for Father's Day. He seemed to think that it would be fun to play games based upon some of the math ideas I had been trying out on him, and he made me a cute little precursor of the Syracuse Algorithm Game. I took my son's idea seriously, as you can see from the games in Chapters 2, 3, and 4 of this book. The 19 games I am including are only a sample of the kinds of games you can make out of the Recreational Number Theory problems introduced in Chapter I. Many of the games described in Chapters 2, 3, and 4 have been played by elementary school teachers in some of my Numberama teacher workshops. Teachers generally found them to be a productive, fun-loving way of helping children learn their arithmetic skills. They also had some excellent suggestions in terms of modifying various aspects of the games (see the section below for their suggestions). However, please keep in mind that my purpose in describing these games is only to offer you a basic framework. It is my hope that you will develop the game ideas for yourself, according to your own unique needs, interests, imagination, and artistic capabilities. Lastly, it is important to keep in mind that the theme for all of the games in Chapters 2, 3, and 4 and all of the problems in Chapter 1, is that the children should be having fun while they are learning mathematics. The game equipment that I have used are gameboards, dice, number cards, play money—in all denominations from \$1 to \$1,000, Lego® pieces for the players, and game rules. Chapters 2, 3, and 4 are divided according to the required arithmetic skill levels for children to play these games.

GAME IDEAS FROM TEACHERS AT NUMBERAMA WORKSHOPS

Some ideas that came out of my Numberama teacher workshops in regard to the games are as follows:

1. Form teams of two or more children.
2. Put a time limit on how long a child can take to give an answer.
3. Use an attachment to the gameboard instead of separate number cards.
4. Give a child at least some money even when an answer is incorrect.
5. Make the games colorful and artistically attractive.
6. Include paper with numbered items for a child to keep a record of.
7. Instruct all children to work on the problem while a child has a turn.
8. Encourage children to make exchanges of money in the game.

I find all of these ideas to be excellent suggestions, and I'm sure you will have more of your own as you read through Chapters 2, 3, and 4. My own suggestion is to play the game after the children have had a chance to explore the ideas that the game is based upon. In other words, I believe the games will be most effective when the children have already experienced the discovery processes described in Chapter 1.

CHAPTER 1

Recreational Number Theory Problems

Abstract: Chapter 1 comprises the nuts and bolts of Numberama, as it includes all the problems that I have included in Recreational Number Theory as part of my Numberama program. Each problem has the designated skill level required, and the problems begin with the Subsets and Circle problem, which utilizes only addition and subtraction, though a knowledge of basic algebra would be required for middle school or high school students to understand the algebraic formulations of this problem. Early in the sequence of problems I go to the Perfect Numbers problem, which admittedly is my favorite problem, and requires a knowledge of multiplication and division, along with once again a knowledge of basic algebra for the same purpose as the Subsets and Circle problem. The learning and teaching strategies that I have included in the Perfect Numbers problem are especially effective in awakening students to the mystery, surprise element, and beauty inherent in our number system, as well as developing an understanding of mathematics as an exciting open-ended field of study with unknown problems that can be worked on with high level computers. Later on in my sequence of problems, one encounters the Sums of Squares problem, which goes back to Fermat, one of the original founders of Number Theory, and once again is an excellent way to develop an appreciation in students of the beauty and mystery involved in our number system. Taken as a whole, these problems from Recreational Number Theory that I have chosen to utilize in my Numberama program can serve as the "magic" needed to demonstrate the inherent joy of mathematics to our students in the school system—and to people of all ages.

Keywords: Abundant numbers, Circles, Clock arithmetic, Deficient numbers, Farey fractions, Fibonacci numbers, Goldbach conjecture, Kaprikar numbers, Linear diophantine equations, Magic squares, Multiplicative persistence, Pascal's triangle, Perfect numbers, Powerful numbers, Prime numbers, Semi-Prime numbers, Subsets, Sums of squares, Triangular numbers, Weird numbers.

1. SUBSETS AND CIRCLES (A, B, J)

This first problem is appropriate for all ages, kindergarten through college. As with most of the problems, it can be done either from the elementary level of arithmetic or from a more advanced perspective of first-year algebra. I recommend a series of three stages in this problem, as follows:

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Stage 1

Define a set as a collection of objects: a set of chairs, set of people, set of numbers, *etc*. Describe the set of numbers as {1,2}, {2,4,6}, {2,5}, {6}, *etc*. Define a subset as part of a set; thus {1,2} is a subset of {1,2,3}; {2} is a subset of {2,5}, but {2,4} is not a subset of {2,5}. Include both the whole set and the empty set Ø—consisting of no elements—as subsets of every set. Thus, all the subsets of {1,2,3} are {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}, and Ø. Have the children make a table like the following (Table **1**):

Table 1. Stage 1a of subsets & circles problem.

Number of Numbers	Number of Subsets
3	8

Depending upon the age range you are dealing with, work out the subsets with the children for sets up to three or four numbers, and have them do the next set of numbers on their own. You will find the pattern doubles in the following way (Table **2**):

Table 2. Stage 1b of subsets & circles problem.

Number of Numbers	Number of Subsets
0	1
1	2
2	4
3	8
4	16
5	32

Ask the children to say what the pattern is and guess how many subsets they think would be in a six-element set. For grades 4 through 6, ask the children how many subsets would be in a 10-element set, a 12-element set, *etc*. Ask algebra students to determine the algebraic formula that describes the number of subsets in any set of **n** elements. Hint that it has something to do with exponents. The answer is that for any number n, the number of subsets in a set of **n** numbers is 2^n, as can easily be experimentally verified by Table **2**.

Stage 2

Now that the basic idea of experimenting with numbers and finding a pattern has been established, the children are ready for a true adventure in mathematics.

Draw a circle with two points—spaced equally apart—on it and connect the two points with a line—in the following way (Fig. **1**):

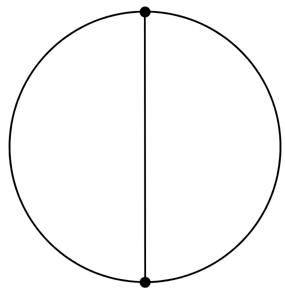

Fig. (1). Stage 2A of Subsets & Circles Problem.

Make a chart like the following:

Number of Points	Number of Regions

The children will observe that when there are two points on the circle, the circle is divided into two regions. Continue this process until five points on the circle are reached—always spacing the points equally apart. The circle will look like the following (Fig. **2**):

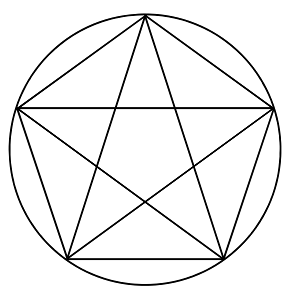

Fig. (2). Stage 2B of Subsets & Circles Problem.

The children will undoubtedly observe that the number of regions is doubling as the number of points is increased, as can be seen from the following table (Table **3**):

CHAPTER 2

Games of Recreational Number Theory: Skill Levels Through Addition, Subtraction, and Multiplication

Abstract: Chapter 2 consists of the games that require the skill levels through addition, subtraction, and multiplication. These games consist of The Tri Game, The Fib-Tri Game, The Multiplicative Persistence Game, The Tri-Square Game, The Tri-Squar--Cube Game, and The Kaprekar Number Game. These games are in general appropriate for children in grades 1, 2, and 3, depending on the children's arithmetic skills level. However, they can be used as enjoyable recreational games for children of all ages, inclusive of mathematically gifted children. See the sections "Introduction to the Games" and "Game Ideas from Teachers at Numberama Teacher Workshops" in the beginning of this book, and **some of the teacher and teacher education Numberama workshop participant responses in the Appendix f**or relevant information about playing these games effectively with children, as well as how these games were developed.

Keywords: Recreational number theory games, The Fib-Tri game, The kaprekar number game, The multiplicative persistence game, The Tri game, The Tri-square game, The Tri-square-cube game.

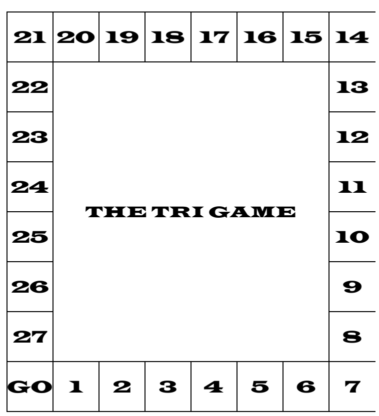

Fig. (9). The tri game (b).

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RULES FOR THE TRI GAME (FIG. 9)

1. Each player tosses die to determine order of turns and order of choice of banker.
2. Player tosses die and lands on number space. He/she must say the appropriate Triangular number. The Triangular numbers are the numbers in the sequence 1, 3, 6, 10, 15, 21, 28, 36, ... obtained by adding one more than the difference of the previous two numbers to the last number. The next number in the sequence would be 45 since $36 - 28 = 8$, $8 + 1 = 9$, and $36 + 9 = 45$. Thus if a player lands on 9, he/she must say the 9th Triangular number, which is 45.
3. The player whose turn is next looks at the back of the number card to see if the original player is correct. If he/she is correct, the original player gets that amount of money, *i.e.*, the Triangular number, from the bank.
4. If a player lands on GO exactly, he/she receives \$100 and the game ends. If a player passes GO, he/she goes through the procedure for the number space he/she landed on; then the game ends. The player with the most money is the winner.

Number Cards for the Tri Game

1	1
2	3
3	6
4	10
5	15
6	21
7	28
8	36
9	45
10	55
11	66
12	78
13	91
14	105
15	120
16	136
17	153
18	171
19	190

Contd.....

20	210
21	231
22	253
23	276
24	300
25	325
26	351
27	378

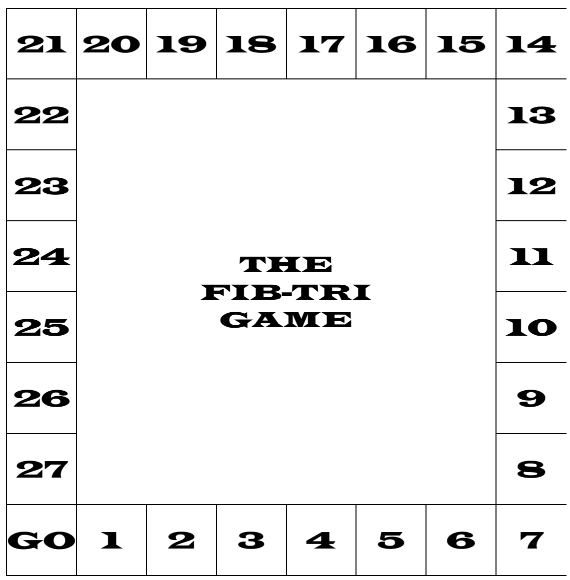

Fig. (10). The fib-tri game (b).

RULES FOR THE FIB-TRI GAME (FIG.10)

1. Each player tosses die to determine order of turns and order of choice of banker.
2. The player tosses die and lands on number space. He/She must say the appropriate Triangular number and Fibonacci number. The Triangular numbers are the numbers in the sequence 1, 3, 6, 10, 15, 21, 28, 35, ... obtained by adding one more than the difference of the previous two numbers to the last number. The next Triangular number in the sequence would be 45, since 36 − 28 = 8, 8 + 1 = 9, and 36 + 9 = 45. Thus, if a player lands on 9, he/she must say the 9th Triangular number is 45. The Fibonacci numbers are the numbers in the sequence 1,1, 3, 5, 8, 13, 21, 34, ... obtained by adding the sum of the previous

CHAPTER 3

Games of Recreational Number Theory: Skill Levels Through Multiplication and Division

Abstract: Chapter 3 consists of the games that require the skill levels through multiplication and division. These games consist of The Prime Number Game, The Perfect Number Game, The Semi-Perfect Number Game, The Powerful Number Game, The Divisor Game, The Sum of Squares Game, The Syracuse Algorithm Game, and The Fib-Tri-Prime Game. The games in Chapter 3 are in general appropriate for children in grades 4, 5, and 6, depending on their skills levels, inclusive of gifted children. The Perfect Number Game has been especially popular with children, in particular because of the stimulating open problems about perfect numbers in mathematics; *i.e.* are there infinitely many perfect numbers and does there exist an odd perfect number (see the Perfect Number problem section in Chapter 1). The games in Chapter 3 are excellent teaching devices to motivate children to practice their multiplication and division skills in an enjoyable way, in addition to enhancing their creative thinking capacities (see some of the teacher and teacher education Numberama workshop participant responses in Appendix).

Keywords: The divisor game, The Fib-Tri-Prime game, The perfect number game, The powerful number game, The prime number game, The semi-perfect number game, The sum of squares game, The syracuse algorithm game.

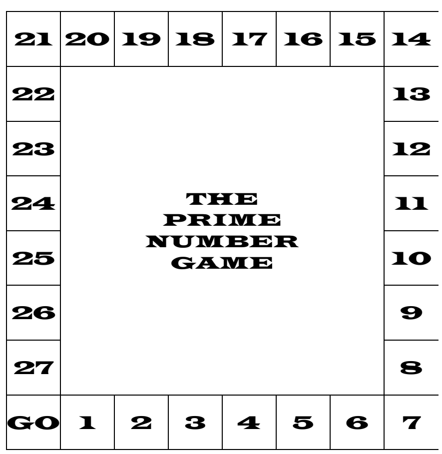

Fig. (15). The prime number game (g).

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RULES FOR THE PRIME NUMBER GAME

1. Each player rolls die to determine order of turns and order of choice of banker.
2. Player rolls die and lands on number space. He/she must say the corresponding prime number (P), twin-prime number (T), and semi-prime number (S) (see the Definitions below). For example, if a player lands on 5, he/she must say the fifth prime number, the fifth twin-prime number, and the fifth semi-prime number.
3. The player whose turn is next looks at the back of the appropriate number card to see if the original player is correct. If the player is correct, he/she must multiply the sum of all the digits in the three numbers he/she landed on by the number space. If he/she is again correct, he/she gets that amount of money from the bank.
4. If player lands on GO exactly, he/she receives $100 and the game ends. If player passes GO, he/she goes through the procedure for the number space he/she landed on; then the game ends. The player with the most money is the winner.

DEFINITIONS FOR THE PRIME NUMBER GAME (FIG.15)

Prime Number: a number that is only divisible by the number itself and 1. Examples of prime numbers are 3, 7, 11, 17, 19, 29.

Twin-Prime Number, prime numbers that differ from another prime number by either 1 or 2. Except for 2 and 3, all twin-prime numbers differ by 2 from another prime number. Examples of twin-prime numbers are 3, 5, 7, 11, 13, 17, 19, 39, 31, 41, 43.

Semi-Prime Number, numbers that have exactly 4 divisors, including the number itself and 1. Examples of semi-prime numbers are 8, 15, 21, 33, 35, 62.

Number Cards for the Prime Number Game

1	P2	T2	S6	$3
2	P3	T3	S8	$6
3	P5	T5	S10	$12
4	P7	T7	S14	$16
5	P11	T11	S15	$30
6	P13	T13	S21	$36
7	P17	T17	S22	$42
8	P19	T19	S26	$48
9	P23	T29	S27	$54

Contd.....

10	P29	T31	S33	$60
11	P31	T41	S34	$66
12	P37	T43	S35	$72
13	P41	T59	S38	$78
14	P43	T61	S39	$84
15	P47	T71	S46	$90
16	P53	T73	S51	$96
17	P59	T101	S55	$119
18	P61	T103	S57	$126
19	P67	T107	S65	$133
20	P71	T109	S69	$140
21	P73	T137	S74	$147
22	P79	T139	S77	$154
23	P83	T149	S82	$161
24	P89	T151	S85	$168
25	P97	T171	S86	$175
26	P101	T181	S87	$208
27	P103	T191	S91	$216

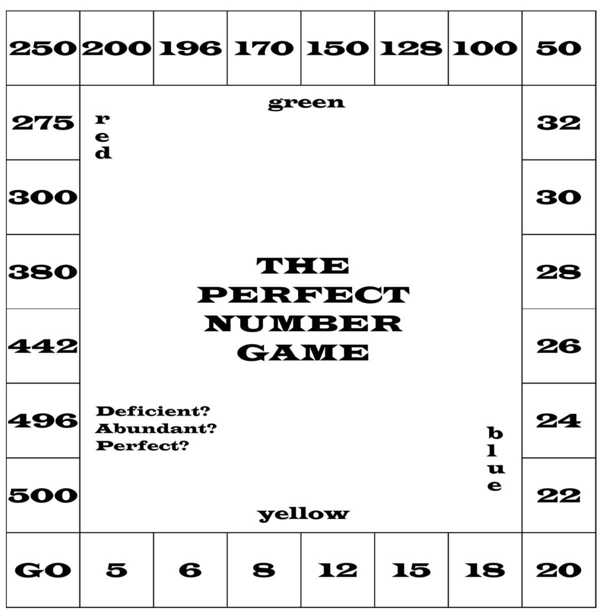

Fig. (16). The perfect number game (g).

CHAPTER 4

Games of Recreational Number Theory: Skill Levels Through Division and Fractions

Abstract: Chapter 4 consists of the games that require all the arithmetic skill levels through fractions. These games consist of The Clock Arithmetic Game, The Pascal's Triangle Game, The Anomalous Fractions Game, The Farey Fractions Game, and The Numberama Game. The games in Chapter 4 are appropriate in general for children in grades 6, 7, and 8, depending on their skill levels, inclusive of gifted children. These are the most challenging games in the set of Numberama Recreational Number Theory games in this book. These games can be especially stimulating for gifted children, and are an excellent way for middle school children to practice and improve upon their skills with fractions (see some of the teacher and teacher education Numberama workshop participant comments in Appendix). The Numberama Game is an excellent teaching device to practice virtually all of the Numberama Recreational Number Theory problems in Chapter 1 of this book.

Keywords: The anomalous fractions game, The clock arithmetic game, The farey fractions game, The numberama game, The Pascal's triangle game.

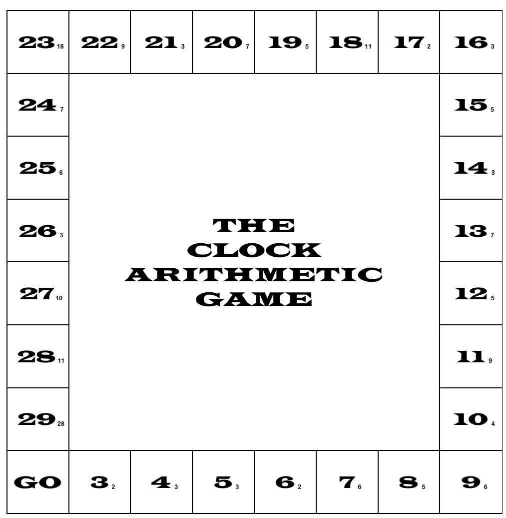

Fig. (23). The clock arithmetic game (h).

Elliot Benjamin

RULES FOR THE CLOCK ARITHMETIC GAME (FIG.23)

1. Each player rolls die to determine order of turns and choice of banker.
2. He/she uses clock with the big number of hours and must try to find a number on the clock that when multiplied by the small number gives the number "1" on the clock; this number is called the "multiplicative inverse" of the small number. For example, number space 11_9 means an 11-hour clock and 9 x 5 = 45 but going around the clock 4 times yields 1 since 11 x 4 = 44 and 45 = 44 + 1, so 5 is the multiplicative inverse of 9 on an 11-hour clock: we will write 45 ≡ 1 mod 4.
3. It may happen that there is no number which, when multiplied by the small number, yields 1. Look at 6_2. For example, 2 x 1=2, 2 x 2 = 4, 2 x 3 = 6, 2 x 4 = 8 ≡ 2 mod 6; 2 x 5 = 10 ≡ 4 mod 6, 2 x 6 = 12 ≡ 6 mod 6, and the process just repeats. Two, therefore, has no mulitplicative inverse on a 6-hour clock.
4. Player must say what the multiplicative inverse is of the small number on the big number clock—or state there is no multiplicative inverse if this is the case.

For another example, take 8_5 = 5 x 5 = 25, 25 = 24 + 1, and 24 = 8 x 3 goes around an 8-hour clock 3 times so 25 = 1 mod 8; 8 is therefore the multiplicative inverse of 5 on an 8-hour clock.

5. Player whose turn is next looks at back of number card. If original player is correct that there is no multiplicative inverse, he/she gets the amount of money from the bank of the big number on the number space. If the player gives the correct multiplicative inverse, he/she must multiply the multiplicative inverse by the clock number (big number). For example, on number space 11_9, since the multiplicative inverse of 9 is 5, player would get 11 x 5 = $55 from the bank.
6. If player lands on GO exactly, he/she gets $100 and game ends. If player passes GO, he/she does procedure on number space, then game ends; player with the most money wins the game.

Number Cards for the Clock Arithmetic Game

3_2	2 x 2 = 4 ≡ 1 mod 3	2 x 3 = $6
4_3	3 x 3 = 9 ≡ 1 mod 4	3 x 4 = $12
5_3	3 x 2 = 6 ≡ 1 mod 5	2 x 5 = $10
6_3	No	$6
7_6	6 x 6 = 36 ≡ 1 mod 7	6 x 7 = $42
8_5	5 x 5 = 25 ≡ 1 mod 8	5 x 8 = $40
9_6	No	$9
10_4	No	$10
11_9	9 x 5 = 45 ≡ 1 mod 11	5 x 11 = $55

Contd.....

12_{5}	5 x 5 = 25 ≡ 1 mod 12	5 x 12 = $60
13_{7}	7 x 2 = 14 ≡ 1 mod 13	2 x 13 = $26
14_{3}	3 x 5 = 15 ≡ 1 mod 14	5 x 14 = $70
15_{5}	No	$15
16_{3}	3 x 11 = 33 ≡ 1 mod 16	11 x 16 = $176
17_{2}	2 x 9 = 18 ≡ 1 mod 17	9 x 17 = $153
18_{11}	11 x 5 = 55 ≡ 1 mod 18	18 x 5 = $90
19_{5}	5 x 4 = 20 ≡ 1 mod 19	4 x 19 = $76
20_{7}	7 x 3 = 21 ≡ 1 mod 20	3 x 20 = $60
21_{3}	No	$21
22_{9}	9 x 5 = 45 ≡ 1 mod 22	5 x 22 = $110
23_{18}	18 x 9 = 162 ≡ 1 mod 23	9 x 23 = $207
24_{7}	7 x 7 = 49 ≡ 1 mod 24	7 x 24 = $168
25_{6}	6 x 21 = 126 ≡ 1 mod 25	21 x 25 = $525
26_{3}	3 x 9 = 27 ≡ 1 mod 26	9 x 26 = $234
27_{10}	10 x 19 = 190 ≡ 1 mod 27	19 x 27 = $513
28_{11}	11 x 23 = 253 ≡ 1 mod 28	23 x 28 = $644
29_{28}	28 x 28 = 784 ≡ 1 mod 29	28 x 29 = $812

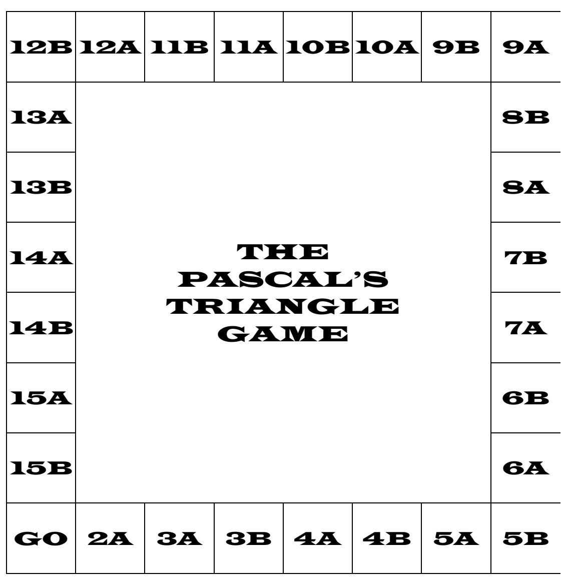

Fig. (24). The pascal's triangle game (h).

Appendix 1: Participants' Responses to Numberama Program

APPENDIX 1 ABSTRACT AND KEYWORDS

In this Appendix 1, I describe a variety of participants' responses to my Numberama program. The participants that I have worked with in my Numberama program include regular elementary school and middle school students, gifted elementary school and middle school students, elementary school teachers, elementary school parents, elementary school workshop participants, gifted children's workshop participants, children at a mental hospital, retirement home residents, various family members and friends, and college students in my introductory mathematics and psychology classes. Unfortunately, I do not have written records of the responses to all my Numberama activities. However, the following sample of participants' responses that I am including in this Appendix is I believe a testimony to the tremendous value of my Numberama program. From my lifelong experience of engaging Numberama with children, I conclude that the greatest benefit of Numberama is for gifted elementary school children in grades 4 – 6, though as can be seen from the student and teacher responses below, Numberama has also had much value for children in regular elementary school classes, especially in grades 5 and 6. I have also learned, as described below, that Numberama has much value in college teaching, both in teacher education classes and in liberal mathematics classes. The following sections in part of the Appendix describe participants' responses to Numberama in a number of diverse settings.

Appendix 1 Keywords

1. Gifted children
2. Mathworks
3. Finite Math
4. Teacher education classes
5. Family Math
6. Retirement homes
7. Elementary school regular classes

NUMBERAMA AT AN ELEMENTARY SCHOOL AS ENRICHMENT BY CHOICE: 2007

Overview

In 2007 I was employed as a mathematics consultant to work in a school system in Maine with 4th, 5th, and 6th grade students who chose to engage in math enrichment activities. I presented various Numberama Recreational Number Theory problems described in Chapter 1 in this book for 15 one hour sessions every other week, and in one session the children played some of the Numberama games in Chapters 2, 3, and 4. The responses of the children were exceptionally enthusiastic, as they welcomed Numberama as a refreshing change of pace from their usual mathematics lessons. Furthermore, a number of them expressed that they would have liked to have Numberama more often than every other week and for longer sessions, as they would forget material from one session to the next. The following are the responses I received to two questions: 1) What did you like most about the program? and 2) What would you change about the program?

What Did You Like Most About the Program?

–I liked the whole thing.
–I liked learning new things because it helped me in my normal class. . . like the factor tree.
–I how we talk about things and he lets us try to find the patterns before he tells us. I think it is worth paying for.
I liked the people in it. I liked how I got to learn more ways to do my mathematics and I liked what we did in Numberama.
–I like the game that we played and next year have more games please.
–I thought it was pretty cool that you could do such a simple pattern to get an accurate answer.
–I LOVE everything. It's really fun being here with all these kids and doing math! I love math. My friends think that I am crazy that I do. I was K. . . 's math partner and we both went to Numberama. It helped us a lot in our other math and made it quicker and easier.
–I liked the part of this where we learned about prime numbers and when we played the circle game.
–I liked it because it gave me a chance to learn something new.
–I really like math and I really liked the challenge of the Numberama program. I think that it should be continued every year. It's really worth the money.
–I like how we talk about different things and learn different things.

What Would You Change About the Program?

–I wouldn't change anything.
–More games.
–I think we should have a longer time or on more days.
–I would change that we have it every week just like we have our specials. I think the 5^{th} and 6^{th} graders should do it together then after the 3^{rd} and 4^{th} could to it together.
–I wouldn't change anything really about Numberama. I would only say that we should have a little more time.
–Keep it the same.
–I think we should do Numberama longer.
–I do wish the classes were closer together because it is hard to remember what we did last class.
–I think the classes should be longer and they should be more than every other week. Other than that I really like the class.
–I wish we could have a group of 5^{th} and 6^{th} grades so it's a little more challenging and one for 3^{rd} and 4^{th} grades so it's a little bit easier and not as challenging.

NUMBERAMA WITH FAMILY AND FRIENDS: 2004 – 2006

Overview

After my three year foundation grant funding for my Numberama program ended in 2004 (see below for a description of teacher responses to my Numberama program during this time), I secured funding from a few different local Maine school districts to continue engaging elementary school students with my Numberama problems and games. During this time period, from 2004 to 2006, I also worked privately with an 8-year-old mathematically gifted student, gave a perfect number lesson to my Creativity in Dementia author friends, and conducted a Family Math evening as a preview to my Numberama work at one of the schools that would be funding my program. In what follows, I am including a letter supporting my Numberama work with gifted children from the mother of the mathematically gifted child I worked with, who happens to also be the child that I gave the fictitious name of Edward to in my essay in Appendix 3. I am also including a letter supporting my Numberama work at a retirement home, from my Creativity in Dementia author friends. Finally, I am including a description of the very positive and appreciative responses that I received from my 2004 Family Math evening.

Letter from Parent of Mathematically Gifted Child: I am writing to express my unqualified support for Elliot Benjamin's Numberama program. Elliot taught my young son last year using his Numberama program. My son loved the experience. He looked forward each week to the day that he got to do math with Elliot and in between meetings delightedly worked independently on the projects and concepts that Elliot had introduced to him. At one point he asked Elliot if they could meet every day instead of once a week—a testament to how engaged my son was in Elliot's teaching.

I was impressed by Elliot's manner in working with a young child. He was patient and gentle yet disciplined and firm when necessary. Elliot Benjamin's creative Numberama program gives children a fun and engaging experience with numbers in which they learn fundamentals of mathematics along the way. I highly recommend his program and am happy to answer any questions. Please feel free to contact me if you wish.

Letter in Support of Numberama with Seniors from Creativity in Dementia Authors/Friends

Elliot Benjamin has a gentle, caring and connective way of communicating with seniors through his Numberama project. This project invites people to challenge and explore numbers in a safe and fun atmosphere. This project offers an innovative and mind-sharpening program for seniors. Numberama offers a group activity that goes beyond the typical retirement and nursing home programming—enticing intellect and creating group energy.

Family Math Night Numberama Responses

Were these family night activities helpful to you? If so, would you tell us how?

18—Yes 1—No

–mentally stimulating
–informative
–family togetherness
–playful approach to math
–understand what my son is doing in school
–tells us about new things that we didn't know
–reviewing patterns is always good
–great for the kids to realize that math can be fun
–it got my mind working and thinking again
–tricks to help me in school
–fun looking at patterns
–patterns are very helpful and thus time-saving
–easy to understand the concepts
–I found out math is even more confusing than I thought
–relaxing, but challenging
–learned an easier way to divide
–learned new things
–fun way of learning different kinds of math that I would not have at school

What did you learn that will be helpful to your children with their schoolwork in the future? Did the activities increase your interest and your children's in math?

–exploration of possibilities—push beyond right/wrong
–patterns to figure out perfect numbers
–encourage to try and keep trying
–simple ideas lead to patterns
–patterns are fun for everyone
–perfect numbers, edges, points,
–focus and have fun
–gave a new way to look at numbers and patterns
–algebra terms
–he'll have to do his own homework
–different games to help teach skills
–refreshed my memory in math
–look for patterns
–tricks and fun stuff
–I have always been interested in math and hope my child's interest will increase more
–look for patterns and concepts in the math problem

NUMBERAMA AT AN ELEMENTARY SCHOOL IN REGULAR CLASSES: 2001 – 2004

Overview

For the four school years from 2001 through 2004, I engaged my Numberama program with elementary school students in grades 3 through 6 at Mt. View Elementary School in Maine;

from 2002 through 2004 my program received grant funding and I visited various classrooms on a regular basis once a week, after voluntarily visiting a few classrooms once a month in 2001. The following teacher responses describe the stimulating and beneficial effects that my Numberama program had on the children in these classes.

Teacher #1: Dr. Benjamin's Numberama Program was a great supplement to our district's math program. Throughout the school year, I often noticed students transfer what they learned about number patterns with Dr. Benjamin to their everyday work in math. Dr. Benjamin was successful at engaging the majority of my students in experiences requiring hypothesizing and testing predictions.

Teacher #2: Dr. Benjamin's Numberama program challenged my students to use what they knew in order to work beyond traditional mathematics. They were always able to quickly pick up from where they left off on his previous visit. Students enjoyed finding perfect numbers.

Teacher #3: "Numberama," presented by Dr. Benjamin was of great value Grade 3/4 students. The children learned about patterns, mathematical terminology, and "thinking like" a mathematician. Most importantly, the students learned that "Doing Math" can be fun and can become a life-long recreational activity!

Teacher #4: The numberama program taught by Dr. Benjamin has been fun for the students in grade five. The students are challenged to think differently about math solutions. Their work has demonstrated that they can do more sophisticated math then they expected of themselves. The challenge and the motivation resulting from their own personal success has been good for them.

Teacher #5: This letter is written in support of the grant submitted by Dr. Elliot Benjamin. Dr. Benjamin has voluntarily worked with my sixth grade math students once a month. The activities he does with my students offers them an enrichment experience that I would not be able to provide my class any other way.

The students are interested, actively involved, and challenged by the activities he chooses to do with them. It would of great benefit to my incoming students to continue the work that Dr. Benjamin started with them in fifth grade.

Teacher #6: This letter is in appreciation of the efforts of Dr. Elliot Benjamin in visiting my 5th grade classroom to work on number theory, sets and other mathematical concepts. All these concepts fit in with what we are trying to cover in our classrooms anyway. In working on sets, my students grew familiar with concepts that have helped them understand division, fractions, percents, and probabilities.

His activities dealing with prime and perfect numbers have added greatly to their understanding of division, multiplication, greatest common factor and possibly other areas yet to come. The students really enjoy "playing" with numbers, and they also appreciate hearing a different voice and style of teaching. All in all, the program is very worthwhile, and I wish they could have the benefit of this expertise once a week instead of once a month. It is a well-spent hour in our program.

NUMBERAMA IN MATHWORKS WORKSHOPS: 1990 – 1993

Overview

The Mathworks (Numberama) workshops that I conducted in Belfast, Maine from 1990 to 1993 was designed primarily for elementary school and middle school teachers actively engaged in teaching mathematics. The responses of the teachers were generally quite appreciative and enthusiastic. In the two student responses that follow, the first student was an adult GED teacher who found my Mathworks/Numberama problems to be useful not only for the more advanced students, but also for the slower students who needed more practice in the basics.

The second student went way beyond anything I could have expected in a Mathworks participant's responsiveness, as she included a detailed description of how she applied various Numberama problems that I presented in her three levels of Gifted students that she worked with, including a number of informative attachments about Magic Squares history and constructions, and teaching techniques that she utilized. However, I am not including her Magic Squares attachments, since what is most relevant here is to see how a teacher working with gifted students responded to my Mathworks/Numberama program.

Student #1: I was very pleased to be allowed to take the Mathworks courses. Since I teach GED for adult ed. I come into contact with many adults with little knowledge of basic math and who are intimidated by math.

What I learned from the class is there is a way to make math fun but also give students a way to practice some basic skills while having fun. The concepts presented offer the slower students needed practice in basics but also offers the brighter students a challenge by figuring out whether there was a pattern involved in finding answers.

The two things that I could use in my classes are the information about the sums of squares and the Farey fractions.

Student #2: The Mathworks workshop is an ideal mathematics course for teachers of elementary math students, and especially for those who work with advanced math students. G/T teachers, working as they often do outside the classroom with small study-groups ranging in size from four to eight students, must provide interesting materials that will motivate gifted learners. The purpose of the small-group format is to enable the teacher to “push” the students, challenging them at the upper limits of their thinking abilities. The content for the specialized group is not the traditional grade-level math curriculum, but a potpourri of math activities that emphasize the conceptual level of mathematics. Following is a brief analysis of ways in which I've used Mathworks topics in my program for students who are mathematically gifted and/or intellectually gifted.

G/T Math, Level 4

Topics most useful at this level were Clock Arithmetic, Fibonacci Numbers, Magic Squares, Pascal's Triangle, and Triangular Numbers.

Given enough preparation time, a teacher can design problems at progressively more challenging levels, of course, to that they can be used at different ability levels. I was fortunate in locating just such a progression of Magic Squares activities. . . . Beginning with the relatively simple Magic Circles, Level 4 students were able to complete Magic Triangles with ease. Magic Stars was much more difficult, even for some Level 5 students.

After completing Magic Squares, I challenged Level 5's with Pure Magic Squares: Create a magic square using the even, consecutive digits 10 – 26. The square in the center must be numbered 18. Note a: Most students thought Magic Square Formula. . . was "too easy." Note b: Many students, even gifted students in Level 6, become frustrated when asked to create their own magic squares. They can do it, but they don't want to/don't enjoy it.

G/T Math, Level 5

Favorites at this level, in addition to Level 4 activities, are Abundant Numbers, Perfect Numbers, and Weird Numbers.

The most successful lesson at any level was the Fibonacci Numbers lesson with Level 5 students. The only explanation I can offer for the enthusiastic response to this lesson is that I introduced it as a "mystery of science,", as something that we know to be true in nature yet do not know how to explain. After giving several examples (pine cones, daisies, sunflowers, star fish, sand dollars, *etc*.), I used the "pairs of rabbits" problem. . . . The students were, in a word, fascinated. This week, I'll learn whether the excitement was enough to cause them to complete a homework assignment and return it a week later.

G/T Math, Level 6

Most students that I work with have not reached Level 6, but I did try an extension of Fibonacci Numbers with advanced students in grade 5 who are intellectually gifted as well as mathematically gifted. Not entirely successful! Beginning with a warm-up activity using domino pieces. . . . and Domino Digs, I then introduced the table shown as Attachment 6. Of six students, only two were interested enough to complete the table as homework and bring it in the following week. Possible explanation: too much of a good thing; "boring" to them because they knew the patterns so well, knew what to expect. . . G/T students have a very low tolerance for repetitive work.

Summary

In summary, this workshop provided a large number of ideas that I can use with my students. This will be evident, I think, after a quick review of Attachment 7, an outline of one of my Level 5 study-groups. Several activities done in the workshop were used directly to meet cognitive and affective objectives listed on this outline, especially cognitive objectives 1 and 2 and affective objectives 1, 2, and 4.

* The "Levels" refer not to grade-levels, but to levels of difficulty. Most study-groups are multi-grade.

NUMBERAMA IN MY TEACHER EDUCATION COURSES AT UNIVERSITY OF MAINE: 1990 – 1993

Overview

I experimented with various problems from Recreational Number Theory that were extracted from Chapter 1 of this *Numberama* book, in the course of my teaching the MAT 107-108 course sequence for future elementary school teachers at the University of Maine from 1990 to 1993. In my first semester as an instructor in this course, I quickly realized that the traditional exam structure for the course, consisting of numerous in-class unit exams, cumulative exams, and a final exam, was perceived as debilitating, oppressive, and laden with fear for practically every student in my class. I made a decision to change the exam structure drastically, instituting three take-home exams plus numerous "special problems" as described in the Numberama Recreational Number Theory problems in Chapter 1 of this book. The responses of the class were for the most part one of great relief and appreciation.

In this section of the Appendix, I am including 16 verbatim responses from students over the course of the six semesters that I have taught these classes. I believe that these accounts speak better than anything I can describe about how my students responded. However, I would like to emphasize a turning point in my approach to teaching the class that occurred during the second year of my teaching this course sequence. To my surprise and disappointment, a number of students found the "special problems" to be rather anxiety producing and frustrating, as they labored for hours upon end to find the solutions to these problems, while panicking that their grade would suffer for not coming up with the correct formulas. Although there was still considerable appreciation for the value of these problems, the conflict and concern about the negative responses motivated me to decide to minimize the required number of these problems in the second semester and offer most of them as bonus problems. For required problems I removed the emphasis on the grade penalties for not correctly solving a problem, stressing that what I was really looking for was an honest attempt to work on the problem.

Although I gave a fewer number of total problems over the semester, I found myself developing the problems more informatively and more extensively in class when I both introduced the problems and subsequently went over them. As can be seen from some of the verbatim accounts that follow, the student responses were quite dramatic in the transformation of their previous anxieties into a playful, appreciative, enthusiasm toward working on these problems over the second semester. Some of the students did research on their on their own in the library, reading more about Number Theory and the mathematicians who originated some of the theories.

A modest number of my students expressed a strong interest in purchasing my *Numberama* book in order to utilize my Numberama techniques of Recreational Number Theory with their own eventual students. However, as can be seen from these student accounts, although the great majority of the descriptions are very appreciative and positive, there were a few students who had much difficulty overcoming the anxiety that they felt toward math, and did not have a positive reaction to working on my Numberama problems.

I believe an important variable that made my presentations more effective the second year I taught the class was my consultant work in the Belfast Area School District, where I was working in a number of fifth and sixth grade classrooms, utilizing the very same Recreational Number Theory problems that I was presenting to my class at University of Maine. Over time the ideas, problems, and techniques gradually took shape in both of my settings, and made me confident that Numberama and Recreational Number Theory has definite concrete value with children—at least from my own experiential perspective.

Student Responses

Student #1: I found your special math problems a challenge. They made me do a lot of deep thinking and figuring. The ability to solve problems is a must in today's world. It's much too easy to use a machine to get the correct answer. With your problems, I had to do all the work using my math skills. It was challenging and frustrating, but at the same time, it was rewarding when I got the correct answer. These have definitely given me a new outlook on math. A challenge is good for everyone and these can be used with kids who need to be challenged to show the joy of math.

Student #2: As a result of the special problems, I've learned that math is not the black and white science that I thought it was. I am a person who thinks in terms of abstracts and possibilities. My confidence in my math abilities has always been low. However, seeing that there are various ways to approach a problem proves that math isn't always an exact right or wrong situation. Solving the special problems has involved creativity and thought. There were no specific formulas. The old perceptions of memorizing formulas and doing busy work haven't held true. It has also been helpful to approach things from a child's perspective, reshaping our own way of thinking. There may be hope for me and math after all.

In addition, rather than taking strategies for granted from rote memory of childhood, we were able to learn why the strategies are true. The methods were explained, answers were proven. More than just the formula and answers—we learned the why. Finally, the background to what we always learned helped make the picture complete in logically understandable terms.

Student #3: I feel that Math 107 has broadened my views on the concept of math. Over the course of the semester, the math curriculum took on ideas such as problem-solving in chapter one and special problems which were assigned weekly. I found the problem-solving extremely difficult, especially at first. However, the class enthusiasm as well as that of the professor, helped to coax me into at least “getting my feet wet” through trying my best with a somewhat more positive attitude. The special problems seemed easy when seen in class, and they sparked a new way of thinking more openly to new mathematical ideas. I admit, some seemed fun and I am now able to visualize enjoyment that students, at varying levels would get from these. Even though I was very hesitant in trying out new ideas, I gained a more open-minded feel of new concepts. Whether or not I may one day pass on these ideas to my elementary class, I have gained further insight, and am perhaps better able to teach from having been exposed to them.

Student #4: When I found out I needed a science or math course to fulfill my degree requirements, I flipped a coin and it turned up heads which meant a math course was on my schedule. My expectation of a study in mathematics was a semester full of fear and dread because I hadn't studied math in a formal way for over thirty years. This fear subsided as prior knowledge was rekindled by the step by step presentation of new methods to problem solving. New knowledge was presented with clarity and systematically that it made problems not as difficult or frightening as I would have found them in past.

Especially helpful were the special problems which we as a class had to grapple with. They were very thought-provoking and at times frustrating until bells and lights of understanding went off leading to the probably solution of the problems. The probable solutions in my case were not always correct. They were, however, near enough to being correct so that when the correct solution was given in class the bells and lights of new understanding would really go off. Insightful learning is often times its own reward.

These problems were also every useful when I substitute taught a sixth grade math class a few times. I presented the special problems of perfect numbers and Pascal's Triangle and discovered they really appreciated the change of pace. The amount of attention and energy most of the students applied to these problems indicated to me that learning math doesn't always have to come in a traditional package.

This class has given me a new perspective of mathematics; that studying math doesn't have to be a fearful or dreadful experience, but is an insightful experience leading to new methods of problem solving.

Student #5: I think the special problems were a worthwhile addition to the material that we covered in this class. Each special problem required logical thinking and a systematic approach to find a pattern that fit the requirements. This helped me to organize my efforts toward a specific goal. They also gave me an insight into how mathematicians develop a theory that applies to all possibilities. It was a challenge to search for a key to these problems and then interpret what was discovered. This took more effort than simply reading a proof in the text but it was also more profitable. I found some of the proof in our text difficult to understand because of the complex way that they were presented. The special problems made the class more interesting but their main value was to create a method of thinking that was not prejudiced by preconceived ideas.

Student #6: My mind is again fuzzy after late nights working on math. Working on Geometry proofs however doesn't give me the satisfaction that long hours working on the special problems do. When I do a Geometry proof it brings back unpleasant memories of High School Geometry and never knowing (or caring) where to start with a proof. If I could look at proofs in the same way as special problems, as a challenge that is exciting and keeps pushing me to find an answer, it would be more fun than drudgery.

It's that challenging aspect that appeals to me. Also, as in the magic square problem, if you do figure out a solution it can spark further interest. I know I felt quite humbled after all the work I put into finding the solution, only to find through research that there were many more ways to solve the problem.

I think, as a future teacher, that the game-like challenge of special problems can be utilized to spur children to increase their learning ability. However—I hope my students won't stay up too many nights until 2 and 3 a.m.

Student #7: By using magic squares in class, I feel I can motivate children. First of all we could do one on the board as a class. I would do the 3. Then I would encourage the children to look for a pattern (horizontal, vertical, and diagonal—all equal 15). I'd ask if they noticed anything else, for example, 3 x 3 = 9, there are 9 squares total.

Then I'd break the class up into cooperative groups to do a 4 magic square. As I want unity in the class, whichever group got the solution could choose a special activity, *i.e.*, reading by teacher for the last 15 minutes of class.

I would encourage the children to work on this during a couple of math classes and at home. If they come up with a magic square, I'd then ask them to record any patterns they find and share these with the class. Extra credit will be given to the class for observations.

I figure this project could be part of math classes during a week's time. Hopefully someone will do a 5 (or more) magic square. If not, I plan to show 5 and 7 and then have the students record and share their observance and then I will share what my college classmates came up with. I'd also do more research on my own beforehand, looking for other help.

Since I've started this class, I have been checking out the bookstores for math books. I'm always on the outlook for some tidbit that might entice a child to probe further, which is what you have certainly done with our class.

Student #8: I really wish I could say I enjoyed the special problems, but I didn't. In our first class you spoke a little bit about math anxiety. Needless to say I suffer from this syndrome. I had to struggle to complete my homework and the added stress of doing a problem that to say the least was difficult was a little more than I felt comfortable with. I can understand your rationale for having us do the special problems. To stretch ourselves, maybe even to put some fun into math, but when you are having a hard time just to understand the basics it's hard to enjoy the recreational side of mathematics.

I did enjoy the format of the class. The reviewing of each week's homework helped me to understand it better. I feel the take-home tests were a benefit to me. If a test is a learning experience, these tests were successful. When I would take the tests, I would spend hours upon hour reading and re-reading the textbook in order to get the correct answers. I feel I learned as much from the tests as I did from any other aspect of the class. If the tests had been given in class, it might have showed that I didn't know instead of helping me to learn what you felt was important for me to know.

I wish I could say that I was looking forward to Math 108, but I'm not. I can say that I know I will learn more about math and it will enable me to teach it better.

Student #9: I found the special problems are very challenging and interesting and sometimes enjoyable—if I am not too overwhelmed for time and don't get too frustrated with them. However, because they were for credit in this course, I found myself devoting a great deal of time using problem-solving techniques to find the answers to the special problems. I would often spend many hours guessing and testing and not come up with a solution to the problem. I would then spend many more hours looking for a pattern that wasn't there, or that I could just not see.

On one particular problem, I threw the book in the corner in frustration and went and soaked in a hot tub full of bubbles in which I searched my brain for a pattern and it came to me while I soaked. However, a lot of time had been spent on this one problem when I finally “saw the light!” Although I was pleased to have arrived at the answer, I was disturbed to think my relaxing bath was interrupted with math homework!

I found myself spending many hours on special problems that I should have spent on my homework and textbook because there were a lot of things I did not understand in class and should have spent more time working on it at home instead of doing the special problems. Maybe we should get credit for doing homework assignments and bonus points for doing special problems.

Student #10: To my understanding the special problems were designed to help develop thinking processes; to help one become aware of the relationships between numbers, formulas, theories; to build curiosity; and to have fun with puzzle-type problems. These appear to be valid objectives.

However, in my opinion, the problems were more than challenging (at least for me) and required hours and hours of figuring. Although I handed in every problem and for the most part, received full credit, I found these problems to be extremely difficult and often wondered how they were benefiting me directly. The grade points earned in no way reflect the amount of time spent on these problems. If such time-consuming and thought-provoking problems are going to be assigned, they should merit more worth. And yet, for those who do not have the hours, fortitude and thinking skills to manage these problems, they should not be penalized.

I suppose that on the average most people enjoy brain-teasers and puzzles and that they enjoy working on such things. I, on the other hand, have a very difficult time with puzzles and such and must work extremely hard to do the simplest kind of puzzles. They are not fun for me, but rather sheer drudgery. Personally, I would have enjoyed the course more without them. However, they represent the nature of math and for those who are inclined, I guess they do build curiosity and make math fun.

I would probably consider these special problems suit your objectives well.

Student #11: I felt the special problems made math interesting. It provided a break from the routine of whatever subject we were on. I think it's a wonderful idea to use with kids; it may spark someone's interest in math. The problems seemed like games or challenges. I also felt pressure was taken off by not making the special problems mandatory. Students could feel positive about them and therefore have more interest in doing them. Some parts of math need to start being fun so that more students will develop an interest. Children can get some positive reinforcement by doing the problems through the system of bonus points.

Student #12: Re: Math 108: I felt the special problems were much less stressful this semester as they were mostly used as bonus and not required for credit. Although I still attempted to solve the problems, I did not feel so bad when I did not find a solution even though I spent a lot of time on it. I did not feel so compelled to spend a great deal of time and frustration finding a solution especially when I drew a mental block as to where to even begin!

In all honest, I have to confess that if I hadn't had last semester to compare it with, I probably would not have spent as much time as I did on the special problems simply because they were not always required but were often considered only as bonuses. My human nature is to do my best of what is required of me only. I still do not consider math as recreational although I do not have the anxiety I had at the beginning of MAT 107.

Student #13: I found that the special problems were very challenging. Sometimes Math can be intriguing and gameish. These problems really helped me see mathematics in a different light. It was fun! I really like the idea that the value of the problems were bonus. The pressure of doing the special problems was alleviated and made it much more enjoyable to do. Number theory is a really neat part of math. I enjoyed seeing the patterns in many of the numbers (Fibonacci, *etc*.). This is something that students can really relate with and at a variety of levels. I am very interested in employing this into my future classroom in order to challenge students and give them more insight into mathematics.

Student #14: My reaction to the special problems has changed since last semester. Before, I had mixed feelings toward the special problems. I think they were interesting and important. I found myself going to the library more to find out who Fermat was or what a Diophantine number was. As a future teacher who will be teaching mathematics, I think it's important to know all aspects of mathematics that include the history of mathematics as well as the content of mathematics.

Although I may not have been successful at some of the special problems this semester, I did get some value out of them by doing a little reading about some of the mathematicians (*i.e.*, Euclid, Fermat, Cioanthus, *etc*.). Our UM library has a good selection of books on number theory. I was up there one night looking for Diophantine numbers, and I started browsing some of the number theory books. This was something I have never done before. This summer I hope to go up and do some free reading.

To conclude, I guess I did get a learning experience out of the special problems. I think they are useful. I think you should encourage reading about number theory in our next MAT 107-108 series. In addition to the special problems, I enjoyed the history of it.

Student #15: RE: MAT 108. It is the opinion of this student that the special problems were presented in a more relaxed manner than in MAT 107 last semester. In MAT 107 it seemed that we were assigned a special problem almost every week. Because of the frequency of the special problems, they sometimes overshadowed our homework assignments. I felt this added to the stress of an already stressful experience, for a student uncomfortable in math.

This semester the special problems were less frequent and in my opinion better presented. It seemed to me that you took more time setting up the groundwork for the special problems.

I've really enjoyed both 107 and 108 and hope that I can someday help students that I will have in the future to understand math as much as you have helped me.

Student #16: For me the special problems have been an excellent way to improve my problem solving abilities.

I have saved each one and hope to use them again in a class of my own some day. I like how they force a person to use basic skills over and over. Sometimes special problems bring unneeded or unwanted anxiety, but since the problems we had were not for core grade but extra credit it was more of a fun challenge.

Special problems also gave me a great feeling of accomplishment when I finally solved it. All in all it has been a great math class.

NUMBERAMA IN MY FINITE MATH CLASSES AT UNITY COLLEGE: 1990 – 1993

Overview

In my first two years as a mathematics professor at Unity College in Maine, from 1985 to 1987, my students were for the most part not responding very positively to my teaching, and my enthusiasm for teaching the standard Algebra and Pre-Calculus courses that I was assigned to teach was not particularly high. It was around this time that I became interested in Recreational Number Theory (see the Acknowledgments and Introduction sections in this book) and started experimenting with what I designated as Numberama problems with my 7-year-old son Jeremy (see the Introduction to the Games section). It was also around this time that I decided to try out these problems by developing and teaching Finite Math at Unity College, which came about from Stephanie Pall (see the Acknowledgments section) conveying to me that my tremendous enthusiasm when I talked about number theory was something that I should find a way of conveying to my Unity College math students, when I told her about my negative student evaluations).

Finite Math is a standard Liberal Arts major math course at colleges, but my inclusion of my Recreational Number Theory (Numberama) problems contributed quite the "non-standard" twist to the course. I continued teaching my Finite Math courses nearly every semester for the rest of my career as a mathematics professor at Unity College, until my retirement in 2006. The student responses to my Recreational Number Theory problems (which I referred as RNT problems) were for the most part very positive, and this enabled me to eventually get promoted to Associate Professor at the college. During the early years of my Finite Math teaching, from 1990 through 1993, I offered students some bonus points for submitting a candid description of their experiences with these RNT (Numberama) problems in Finite Math. The following is the responses I received from seven students in my classes.

Student #1: Elliot, I have honestly found this course to be enlightening. The material you chose and your patience with the class as "we" stumbled through number theories and infinite prime numbers is commendable. And even though I will probably never have the opportunity to use these systems and theories in the real world, I at least now know where they came from. I personally have found the material to be challenging. In fact I can say from personal experience that the people at Tylenol were enjoying the class also. The material was not hard, it just sometimes required some serious thought to bull my way through it.

The way the class is structured I feel was also beneficial, and allowed each student to think about math in a totally different light. Even the group sessions promoted teamwork and created a reservoir of ideas to solve a problem.

I think that there is even the potential for another Finite Math at the 2000 level. This course could include a more in-depth look at the creators (discoverers) of certain systems and theories, and a term paper on a topic that was not covered in the class.

Elliot, I would have written this regardless of the extra points. I do honestly feel that a 2000 level course may be in order. Thank you, Elliot, I did learn something.

Student #2: I found the mathematics class to be a rather enjoyable one. I am not very fond of math, but I enjoyed the subject material in this class. Through this class, I saw that mathematics is not just a field of numbers made for no real purpose. Instead, I noticed a way to enjoy mathematics and to witness the endless field of numbers and formulas. By this I mean that math seems to be an endless array of sets of numbers, formulas, *etc*. Math is no longer a boring subject to me when I reflect on this class. I have learned in a way to enjoy and respect the beauty of numbers.

Student #3: I felt that the class was unique. Being able to work with other students was helpful. Other students gave me more insight on how to accomplish homework assignments. It made me feel good when I understood something and someone else didn't understand it. I was able to help friends understand it better and *vice versa*. The class was interesting in the beginning but somehow you lost my interest toward the end of the class. I think the reason for this was because it started to get extremely complicated. But what was neat was my other friends helped as well as what I learned in the class.

Student #4: Finite Math: Finite Math has been beneficial to me for brain power. At points in my homework. I would become stuck and would have to push through to find the criteria. This style of learning tends to be more exciting because we don't simply go home and do the homework by repeating and practicing what we have already learned. Rather, we have a puzzle or brainteaser to solve.

The classes were almost always upbeat and interesting and the guest teacher was great.

One problem I have was with the glare on the blackboard. The largest concern in many classes was with the speed that you taught. I couldn't keep up at times, and found myself buried in the notes.

I like the dot game that we learned and I actually stumped a student the other day on it. Perfect numbers was good, and I'd like to learn more about probability.

I have a feeling that being exposed to so many math theories will help me in the future, and will give me an advantage when I at least recognize some of what we learned.

I give high marks for style, preparedness, office hours, and especially enthusiasm and interest.

Student #5: I found Finite Math to be enjoyable. It was a laid back atmosphere, where the student could simply try and have fun figuring out the various problems. One criticism that I would make would be to get rid of the book. The lectures that used the book were boring and not much fun. The days that the book wasn't used were much better. The lectures without the book were at least 3^2 times better. I also felt that the class was more interested and responded better on those days.

Student #6: After the first session of this class, I kept returning in order to challenge my own mind as well as gather ideas which I could put to use in my classroom. I found it very stimulating, since the math taught in the sixth grade is not very challenging. I enjoy math and having to think.

I especially enjoyed the session with the games because I will be able to use those with my students at various times throughout the year. Thank you.

Student #7: When you first asked for introductions and reasons for taking the course the connection between “art” and math seemed vague. I come from a scientific family and everyone has a strong artistic background. Science and art are just objective and subjective views of the same world. Math keys in with this neatly and gives order and reason a relevance in art. I've always loved both the honest clarity of numbers and the geometry of nature. This course fitted my bill and whetted my appetite. Thank you.

Appendix 2: Definitions, Examples, Hints

1. **Counting Numbers:** 1, 2, 3, 4, 5 …. continue adding 1 to each number to obtain the next number.
2. **Even Counting Numbers:** 2, 4, 6, 8, 10 …. continue adding 2 to each number to obtain the next number.
3. **Odd Counting Numbers:** 1, 3, 5, 7, 9 .... continue adding 1 to each number to obtain the next number.
4. **Multiples of** 7: 7, 14, 21, 28, 35 .... continue adding 7 to each number to obtain the next number.
5. **Triangular numbers:** 1, 3, 6, 10, 15, 21, 28 …. continue adding one more than the difference of the previous two numbers to obtain the next number.
6. **Fibonacci Numbers:** 1, 1, 2, 3, 5, 8, 13, 21 …. continue adding the sum of the previous two numbers to obtain the next number.
7. **Prime Numbers:** 2, 3, 5, 7, 11, 13, 17, 19, 23 …. prime numbers are numbers that have no divisors other than 1 and the number itself.
8. **Abundant Numbers:** numbers such that the sum of all the number's proper divisors, *i.e.*, all divisors other than the number itself, is greater than the number. An example of an abundant number is 24 since the proper divisors of 24 are 1, 2, 3, 4, 6, 12, and their sum is 28, which is greater than 24.
9. **Semi-Prime Numbers:** numbers that have exactly 4 divisors, including the number itself and 1. An example of a semi-prime number is 15 since the only divisors of 15 are 1, 15, 3, and 5.
10. **Kaprekar Numbers:** two-digit numbers such that their squares have four digits and the sum of the first two-digit number and last two-digit number of the square equals the original number. An example of a Kaprekar number is 45 since $45^2 = 45 \times 45 = 2025$ and 20 + 25 = 45.
11. **Twin-Prime Numbers:** 2, 3, 5, 7, 11, 13, 17, 19, 29, 31, 41, 43, …. twin-prime numbers are prime numbers such that either the consecutive odd number above it is a prime or the consecutive odd number below it is a prime. We will consider 2 to be a twin-prime number since 3 is a prime. An example of a prime number that is not a twin-prime number is 23.
12. **Perfect Number:** a number such that the sum of all the proper divisors of the number equal the number itself. The first (smallest) perfect number is 6 since the proper divisors of 6 are 1, 2, and 3, and they add up to 6. The second perfect number has 2 digits; the third perfect number has 3 digits.
13. **Semi-Perfect Number:** a number such that some combination of the proper divisors of the number add up to the number itself. An example of a semi-perfect number is 36 since the proper divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 18 + 12 + 6 = 36.
14. **Weird Number:** a number that is abundant but not semi-perfect. Most abundant numbers are semi-perfect and therefore not "weird." However, there is one, and only one, two-digit weird number, and one, and only one, three-digit weird number.
15. **Amicable Numbers:** pairs of numbers such that all the proper divisors (excluding the number itself) of one number add up to the other number and *vice versa*. The smallest

pair of amicable numbers are both in the 200s.

16. **Prime Factorization:** means breaking up a number into a product of prime numbers with exponents. Examples of prime factorizations are 150 = 50 x 3 = 25 x 2 x 3 = 5 x 5 x 2 x 3 = 5^2 x 2 x 3; 108 = 54 x 2 = 27 x 2 x 2 = 9 x 3 x 2 x 2 = 3 x 3 x 3 x 2 x 2 = 3^3 x 2^2. Prime factorizations are always unique: *i.e.*, a number can be broken up into prime numbers with exponents in one and only one way.
17. **Powerful Numbers:** numbers such that all primes in their prime factorization have exponents greater than or equal to 2. Some examples of powerful numbers are 8 = 2^3, 9 = 3^3, 100 = 2^2 x 5^2, 432 = 3^3 x 2^4. Note that 8 and 9 are consecutive powerful numbers.
18. **Set:** a collection of objects; in mathematics we will consider sets of numbers and denote it with parentheses. Some examples of sets are {1,2}, {1, 3, 5}, {2, 8, 9, 11}. Ø means the empty set—which has no elements.
19. **Subsets:** subsets are parts of sets. If the original set is {1, 2, 3, 4, 5}, some subsets are {1, 3}, {2}, {1, 2, 4, 5}. Note that the whole set {1, 2, 3, 4, 5} is a subset of itself, and the empty set Ø is a subset of every set.
20. **Number of Subsets:** a set with no numbers, Ø, has 1 subset: Ø; a set with 1 number, {1}, has 2 subsets: Ø and {1}; a set with 2 numbers, (1, 2}, has 4 subsets: {1}, {2}, {1, 2}, and Ø; a set with 3 numbers, (1, 2, 3}, has 8 subsets: {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}, and Ø. Continuing in this way a set with 4 numbers {1, 2, 3, 4} will have 16 subsets. Note the pattern that seems to be unfolding for the number of subsets for a set of numbers.
21. **Sums of Squares:** (for prime numbers) some prime numbers can be written as the sum of two squares. For example, 37 = 36 + 1 = 6^2 + 1^2 ; 29 = 25 + 4 = 5^2 + 2^2. It is also the case that some prime numbers cannot be written as the sum of two squares: for example, look at 3, 7, 19, and 31. There is a way to determine very quickly whether or not an odd prime number can be written as the sum of two squares. Divide the odd prime number by 4; if the remainder is 1 then the number can be written as the sum of two squares, and if the remainder is 3 then the number cannot be written as the sum of two squares. Note that 2 is the only even prime number, and 2 = 1^2 + 1^2.
22. **Number of Divisors:** means all the divisors of a number, including the number itself, and we denote this by the symbol "d". For example, d (20) = 6 since the divisors of 20 are 1, 2, 4, 5, 10, and 20 and there are therefore six divisors of 20. There is a relationship between exponents of the primes in the prime factorization of a number, and its number of divisors. Note that 20 = 10 x 2 = 5 x 2 x 2 = 5^1 x 2^2. Some other examples are d(7) = 2 and 7 = 7^1; d(25) = 3 and 25 = 5^2.
23. **Syracuse Algorithm:** This says to do the following: take a number: if it is odd, multiply it by 3 and add 1 to it; if it is even, take half of it; continue the process. For example, if the number is 7, the sequence would be 7-2-11-34-17-52-26-13-40-20-10-5-16-8-4-2-1. Note that there are 17 numbers in this sequence, as after 1 the sequence would always repeat itself since we would have 1-4-2-1-4-2-1. The first billion numbers have been tested for the Syracuse Algorithm and it has been found that they always eventually yield 1.
24. **Multiplicative Persistence:** given a two-digit number, multiply the digits together; if you get another two digit number, multiply together again; continue the process until

you obtain a single-digit number. The number of steps it takes to get a single-digit number is the multiplicative persistence of the original number. Some examples are: 24-8, so 24 has multiplicative persistence of 1; 45-20-0, so 45 has multiplicative persistence of 2; 97-63-18-8, so 97 has multiplicative persistence of 3. There is one, and only one, two-digit number that has multiplicative persistence of 4.

25. **Anomalous Fractions:** These are proper two-digit fractions such that the second digit of the numerator fraction and first digit of the denominator fraction are the same, and when you cancel out these two identical digits you get an equivalent fraction to the original fraction, as can be seen by reducing both fractions to the lowest terms. For example 16/64 = 1/4 and 16/64 = 8/32 = 4/16 = 2/8 = 1/4 when reduced to the lowest terms.
26. **Pascal's Triangle:** Pascal's Triangle is the following triangle (Fig. **29**):

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Fig. (29). Pascal's triangle in appendix 2: definitions, examples, hints.

It is found by putting 1's on the outside of the rows and adding two consecutive numbers in a row and writing this sum in the middle space of the row below. Note that the counting numbers are obtained as the second diagonal of Pascal's triangle, and the triangular numbers are obtained as the third diagonal of Pascal's triangle: 1. 3, 6, 10, 15, 21, ... One may also note that the sum of the rows of Pascal's triangle are doubling: 1, 1 + 1 = 2, 1 + 2 + 1 = 4, 1 + 3 + 3 + 1 = 8, 1 + 4 + 6 + 4 + 1 = 16, *etc*.

27. **Clock Arithmetic (Figs. 30, 31):** In an ordinary 12-hour clock, we can do arithmetic as follows: 9 + 5 = 14 − 12 = 2, 8 + 7 = 15 − 12 = 3, 5 x 4 = 20 − 12 = 8, 8 x 6 = 48 − (3 x 12) = 12, 5 x 5 = 25 − 2 x 12 = 1. Notice that for most numbers on the clock, you will never get 1. However, 5 x 5 = 1, and we therefore say that 5 has a multiplicative

inverse, namely 5, meaning a number that when multiplied by the original number, yields 1. On a 7-hour clock, every number, other than 7, has a multiplicative inverse, in the following way: 1 x 1 = 1, 2 x 4 = 1, 3 x 5 = 1, 4 x 2 = 1, 5 x 3 = 1, and 6 x 6 = 1.
We can talk about multiplicative inverses on any number clock; there is a simple way of determining whether or not every number on a clock has a multiplicative inverse (see Chapter 1).

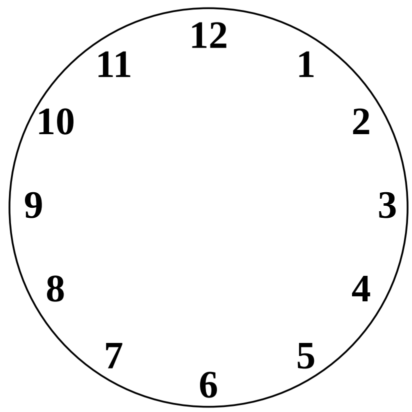

Fig. (30). Clock arithmetic figure #1 in appendix 2: definitions, examples, hints.

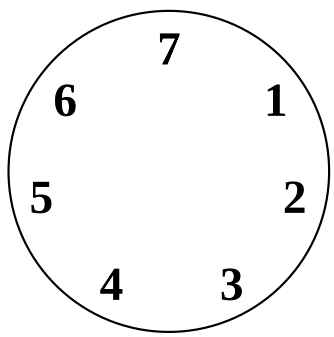

Fig. (31). Clock arithmetic figure #2 in appendix 2: definitions, examples, hints.

28. **Farey Fractions:** these are sequences of fractions of a particular order; the sequence is going from zero to one, is consecutively increasing, all fractions in the sequences are reduced to lowest terms, and the fractions have denominators less than or equal to the order number. The Farey Fractions of orders 1 to 5 are as follows:
Order 1: 0/1, 1/1
Order 2: 0/1, 1/2, 1/1
Order 3: 0/1, 1/3, 1/2, 2/3, 1/1
Order 4: 0/1, 1/4, 1/3, 1/2, 2/3, 3/4, 1/1
Order 5: 0/1, 1/5, 1/4, 1/3, 2/3, 1/2, 3/5, 2/3, 3/4, 4/5, 1/1
Recall that to decide the order of magnitude of two fractions we obtain equivalent fractions using the least common denominator, as follows: 3/5 > 1/2 since 3/5 = 6/10 and

1/2 = 5/10 and 6/10 > 5/10. The distinctive characteristic of Farey fractions has to do with what happens when you cross-multiply two successive Farey fractions, *i.e.* multiply the denominator of the first fraction by the numerator of the second, and then multiply the numerator of the first fraction by the denominator of the second.

29. **Magic Squares:** These are geometric squares divided up into boxes of the same size by constructing horizontal and vertical lines through the square, with a number in each box. The magical property of these squares is that the sum of the numbers in each of the rows, each of the columns, and each of the diagonals, all add up to the same number; this constant sum is called the "magic number." An example of a 3x3 magic square such that all the numbers can be seen from Fig. (**32**):

An example of a 4x4 magic square using all the digits from 1 to 16, such that each digit is used only once, with magic number 34 can be seen from (Fig. **33**):

An example of a 5x5 magic square, using all the digits from 1 to 25, such that each digit is used only once, with magic number 65 can be seen from (Fig. **34**):

There exists a 3x3 magic square, using all the digits from 1 to 9, such that each digit is used only once, with magic number 15. Can you find it?

479	**71**	**257**
47	**269**	**491**
281	**467**	**59**

Fig. (32). Magic squares figure #1 in appendix 2: definitions, examples, hints.

16	**3**	**2**	**13**
5	**10**	**11**	**8**
9	**6**	**7**	**12**
4	**15**	**14**	**1**

Fig. (33). Magic squares figure #2 in appendix 2: definitions, examples, hints.

15	**18**	**21**	**4**	**7**
24	**2**	**10**	**13**	**16**
8	**11**	**19**	**22**	**5**
17	**25**	**3**	**6**	**14**
1	**9**	**12**	**20**	**23**

Fig. (34). Magic squares figure #3 in appendix 2: definitions, examples, hints.

Appendix 3: Numberama and the Perfect Number Game with a Gifted Child

(August, 2008)

I will conclude this book with the following excerpt from an article that I wrote as part of a submission for my 2008 NECGT (New England Conference for Gifted Children) Numberama with Gifted Children workshop.

My article focuses on a very special gifted child, who I will refer to by the fictitious name of Edward, who initiated me into my 2007 - 2008 exceptional experience of teaching *Numberama* in the Old Town, Maine school district.

For many years I have conducted my Numberama program, the name I gave to various Recreational Number Theory problems and games included in my book: *Numberama: Recreational Number Theory in the School System* at a variety of school settings, family settings, community settings, and individual settings with children, adolescents, adults, and seniors. However, in the 2007-2008 school year I had the privilege and unique experience of offering my Numberama program as part of the Gifted & Talented services at the elementary and middle schools in the Old Town, Maine school district, working with between 45 and 50 children for 8 or 9 hours a week, comprised of approximately 15 - 20 children in the elementary school and 25 - 30 children in the middle school. Although I have successfully offered my Numberama program for gifted children in the past, both in schools and with individual students privately, the extensive and intensive Numberama experiences I have had with gifted children in every grade from grades 2 thru 8 this past school year have made me realize the enormous potential for the enhancement of mathematical creativity as well as for the collaborative and productive group participation in gifted children through the use of Recreational Number Theory. As a preview to my NECGT Numberama with Gifted Children workshop, I would like to give a brief illustration of my Numberama work with the gifted child, whom I will refer to by the fictitious name of Edward, who initiated me into my recent exceptional experience of teaching Numberama in the Old Town, Maine school district.

A MATHEMATICALLY GIFTED CHILD

I first encountered Edward in March of 2005 when I was still a mathematics professor at a small college in rural Maine. One of my math professor colleagues asked me if I would be interested in working with the son of a friend of his, a mathematically gifted 5-year-old child, as my reputation for working with children through the mathematical exploration of number patterns described in my self-published book *Numberama: Recreational Number Theory In The School System* was well known in my rural Maine community. Although I was initially reluctant, as my *Numberama* work was geared towards children in grades 3 thru 8 who knew their basic arithmetic skills of multiplication and division, my colleague assured me that Edward was a highly unusual mathematically impressive 5 year old who was able to do multiplication and division of large numbers in his head. Thus my appetite was whetted; I agreed to meet Edward and try out a *Numberama* session with him.

I had worked with a number of mathematically bright children over the years, including children who were in the Gifted & Talented programs in various schools. But I had never encountered anything that even remotely resembled the mathematical abilities of little 5-year-old Edward who was only in Kindergarten at the time, and I am including children in grades 7 and 8 when I say this. Edward could indeed do all kinds of arithmetic in his head, including multiplication and division of numbers way into the thousands, and he even could do calculations with fractions in his head. His mathematical abilities were truly phenomenal, and he loved playing with numbers as much as I do. Edward was ripe for my *Numberama* lessons, and his mathematical creativity was truly astounding. Edward especially liked playing my *Perfect Number Game*, as his mother purchased my *Numberama* book and *Perfect Number Game* from me, and she and Edward would work on my *Numberama* problems and play *The Perfect Number Game* during the week. I would meet with Edward for an hour once a week, and give him suggested number patterns to think about and explore for the rest of the week. Edward became thoroughly immersed in the topic of Perfect Numbers, which is my favorite *Numberama* topic to teach (see my *Numberama* book for an illustration of how I teach this topic). Very basically, a perfect number is a number that is the sum of its proper divisors, *i.e.* you add up all the divisors of a number, not including the number itself, and if these add up to the number then the number is perfect. The first perfect number is 6, since 6 = 1 + 2 + 3, and the second perfect number is 28, since 28 = 1 + 2 + 4 + 7 + 14. Perfect numbers have a wonderful and surprising pattern to them, and there are some unsolved problems about perfect numbers that young children can readily appreciate, such as how many perfect numbers are there? (more than you can count?) and does there exist an odd perfect number? (see *Numberama*).

After discovering the pattern to formulate perfect numbers, we obtained the first five perfect numbers, which are 6, 28, 496, 8128, and 33,550,336 (see *Numberama*). Edward was captivated and intrigued, and I had a mathematical protégé whom I could not stop thinking about. This lasted for ten sessions, at the end of which Edward, who had just turned 6, had assimilated all he was able to of *Numberama* at that time. Summer was approaching, and Edward's mathematical interest was gradually lessening, as the more normal childlike parts of him that enjoyed riding his bicycle and playing outside with other children were taking precedence. We all agreed it was time to take a break from *Numberama*; we parted on very good terms with an openness to what the future might bring for us working together again.

Edward was a somewhat socially shy child, but he enjoyed playing with other children and he was not diagnosed with any kind of psychiatric mental illness that appears in the DSM-IV. The next time I encountered Edward was nearly 2 years later, as I was asked to work with a group of mathematically bright children in Edward's school. Edward was now in third grade (he skipped a grade) and was a few years younger than the rest of the children in the group. It was a group of over 20 very bright children who were mostly in grades 5 and 6, but Edward stood out as by far the child with the highest level of mathematical ability. However, it was apparent that Edward was rather quiet and not very comfortable socially with the other children, while he was bored in the class of children closer to his own age and had trouble with the rigid expectations of his teacher. Although I still would not classify Edward with any DMS-IV psychiatric syndrome, I was able to see that if these social difficulties were to continue, Edward could indeed be in danger of developing psychological problems as he got older. Edward took part in seven of our *Numberama* sessions with the older children, and I also worked with Edward individually for an additional five sessions, one of which was a joint session with a mathematically advanced child of the same age who was being homeschooled. It was apparent to me how much Edward enjoyed having the social and intellectual stimulation of another child whom he could relate to, but I also realized that this other child, although mathematically bright, was not on Edward's mathematical level, and it would not work mathematically to put them together in a group.

The purpose of my individual sessions with Edward was to teach him more traditional math, such as Algebra, in order to keep him stimulated mathematically. However, what Edward really wanted to do was more *Numberama*. In our last session I realized that the abstractness of Algebra was beyond the assimilation abilities of his cognitive 8-year-old mind in the Piaget context, and I yielded to his request to play a number game which involved using the different arithmetic operations (*i.e.*, addition, subtraction, multiplication, division) in creative combinations to come up with a positive number. We spent much of this last session struggling together to try to come up with one of these number puzzle problems that I do believe was an error. At one point when he was frustrated with a different number puzzle, Edward became very sad and tearful, and though I was able to help him work through this, I realized how delicate and vulnerable this 8-year-old artistically inclined mathematically gifted child truly was.

SEQUEL

The sequel to my work with Edward is that he attended a wonderful little private school, Riley school lin Rockport, Maine, which happens to be the same school my own son attended for grades 7, 8, and 9. This school focuses upon nourishing a child's creativity within a social environment full of warmth, caring, and support, containing aspects of A.S. Neill's (1960) Summerhill philosophy and Carl Rogers' (1969) humanistic education philosophy. I think this is a perfect fit for Edward, and in addition to the supportive nurturing atmosphere of his new private school, Edward also has the good fortune of having the loving, caring, intelligent, and dedicated caretaking of his mother.

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