RECENT ADVANCES IN LIFETIME AND RELIABILITY MODELS

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Contents

Foreword

This book is, as far as I have gathered, the first book of its kind. The authors should be commended for spending countless hours researching the literature and explaining in details the connections between many different distributions published during the past decades. I believe there will be so many grateful researchers and readers who will have a broad perspective of all the interesting distributions presented in this manuscript. This book can serve as a foundation for those who are seriously interested in doing research in the field of distribution theory. The content of the book deals with a comprehensive treatment of methods for lifetime models, which has many practical applications in various fields. It seems to me, if one wants to do literature review of the published work in this area, all one needs to do is get hold of a copy of this book. In my opinion, this book is destined to be an extremely important source for motivating the younger researches in the field of distribution theory, in particular lifetime models. This captivated book represents a complete account of important distributions, their properties and their applications in various fields of applied sciences. I am sure that this book will serve as a unique and excellent source of information in the overall field of statistics and probability for many years to come. I admire the effort of the authors to come up with such a fantastic work.

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Preface

The theory of distributions with support in the positive real numbers has grown and matured in the last two decades, becoming one of the main statistical tools for the analysis of lifetime (survival) data. In fact, in many ways, lifetime distributions are the common language of survival dialogue because the framework subsumes many statistical properties of interest, such as reliability, entropy and maximum likelihood.

This book provides a comprehensive account of models and methods for lifetime models. Building from primary definitions such as density and hazard rate functions, the book presents the distribution theory in survival analysis. This framework covers classical methods, such as the exponentiated method, and also the most recent developments in lifetime distributions, such as the beta family and compounding models. Additionally, there is a detailed discussion of mathematical and statistical properties of each family, such as mixture representations, asymptotes, some types of moments, order statistics, quantile and generating functions and estimation. There is also a brief exploration of regression models for the beta generalized family of distributions. Throughout the text, we focus not only on the theoretical arguments but also on issues that arise in implementing the statistical methods in practice. The most recent parametric models in lifetime data analysis are covered without concentrating exclusively on any specific field of application, and most of the examples are drawn from engineering and biomedical sciences. It is important to emphasize that even with omission of some models, the great amount of models available has forced us to be very selective for inclusion in this work. To keep the book at a reasonable length we have had to omit or merely outline certain models that might have been included.

To help readers, lists of notation, terminology, and some probability distributions are given at the beginning of the book. All notational conventions are the same or very similar to the articles from which the models are based. Readers are assumed to have a good knowledge in advanced calculus. A course in real analysis is also recommended. If this book is used with a statistics textbook that does not include probability theory, then knowledge in probability theory is required.

The main five generators of new distributions are grouped into seven sections corresponding to those to which they give names. Chapter 1 contains introductory material with mathematical and statistical background for understanding this book. Chapter 2 deals with the exponentiated method. Explicit expressions for the quantile function, ordinary and incomplete moments, probability weighted moments, cumulants and generating functions are presented for the exponentiated-G family. Chapter 3 discusses the procedure that generates what we call the beta generalized family. Further, useful expansions and several statistical properties are presented. Chapter 4 provides theoretical essays about five special models in the beta family. For each model, its cumulative, density and hazard rate functions have explicit forms and important linear representations, which can be used to obtain some mathematical properties. Two applications are performed in order to illustrate the flexibility of the densities under discussion. Chapter 5 introduces the Kumaraswamy generalized family. In addition, several structural properties are presented and discussed for this family. Among them, useful expansions, quantile and generating functions, moments and mean deviations. Additionally, estimation and generation procedures are investigated. Chapter 6 presents three special cases of the Kumaraswamy generalized family. Some mathematical properties are provided such as the moments and generating function. Useful expansions for the density function and some special cases are presented. Chapter 7 discusses the gamma generalized family proposed by Zografos and Balakrishnan (2009). Several mathematical properties are provided such as expansions for the density and cumulative functions, quantile function, moments, generating function and entropies. A bivariate generalization is presented. Chapter 8 introduces a family of models defined by compounding two (a continuous and other discrete) distributions. We provide important mathematical properties such as moments and order statistics. We discuss the estimation of the model parameters by maximum likelihood and prove empirically the potentiality of the family by means of two applications to real data.

Readership

We hope that this book inspires students that make extensive use of observational data, including finance, medicine, biology, sociology, education, psychology, engineering and climatology. Further, we hope that our readers come to regard this book as a reliable source of information and we gladly welcome all efforts to bring any remaining errors to our attention.

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Conflict of Interest

The authors endorse that the Book content has no conflict of interest.

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Nomenclature

ACRONYMS

AIC	Akaike Information Criterion
BBIII	the beta Burr III distribution
BBS	the beta Birnbaum-Saunders distribution
BBXII	the beta Burr XII distribution
BC	Bonferroni curve
BE	the beta exponential distribution
BFr	the beta Fréchet distribution
BFGS	the Broyden-Fletcher-Goldfarb-Shanno optimization method
BG	the beta-G family of distributions
BGE	the beta generalized exponential distribution
BHC	the beta half-Cauchy distribution
BI	Bonferroni index
BIC	Bayesian Information Criterion
BLa	the beta Laplace distribution
BLN	the beta lognormal distribution
BMW	the beta modified Weibull distribution
BN	the beta normal distribution
BPa	the beta Pareto distribution
BSL	the beta standard logistic distribution
BSPS	the Birbaum-Saunders power series distributions
BW	the beta Weibull distribution
BWG	the beta Weibull geometric distribution
BXIIPS	the Burr XII power series distributions
CAIC	Consistent Akaike Information Criterion
cdf	cumulative distribution function
cf	characteristic function
cgf	cumulant generating function
chf	cumulative hazard function

cmgf	central moment generating function
EE	the exponentiated exponential distribution
EFr	the exponentiated Fréchet distribution
EG	the exponentiated gamma distribution
EGu	the exponentiated Gumbel distribution
EM	expectation maximization
EV	the extreme value distribution
exp-G	the exponentiated-G class of distributions
EW	the exponentiated Weibull distribution
EWPS	the extended Weibull power series distributions
GE	the generalized exponential distribution
GI	Gini index
GoF	Goodness-of-Fit
GW	the generalized Weibull distribution
hrf	hazard rate function
KS	Kolmogorov Smirnov Statistic
LBBS	the log-beta Birnbaum-Saunders distribution
LBW	the log-beta Weibull distribution
LC	Lorenz curve
log-BW	the log-beta-Weibull distribution
LW	the log-Weibull distribution
mgf	moment generating function
MixEW	the mixture of Weibull distributions
MLE	maximum likelihood estimate
mrlf	mean residual life function
pdf	probability density function
PWM	probability weighted moment
rhrf	reversed hazard rate function
sf	survival function
1-MEW	the first modified Weibull distribution
2-MEW	the second modified Weibull distribution
3-MEW	the third modified Weibull distribution

NOTATIONS

$h_\bullet(\cdot)$, $h(\cdot)$, $h(\cdot;\cdot)$	hrf
$S_\bullet(\cdot)$, $S(\cdot)$	sf
$H_\bullet(\cdot)$, $H(\cdot)$	chf
$m(\cdot)$	mrlf
T, Y, X, T^*	Random variables (specifically, T describes failure time)

$\boldsymbol{\theta}, \boldsymbol{\beta}$	parameters vectors
$L(\boldsymbol{\theta})$	likelihood function
$\ell(\boldsymbol{\theta})$	log-likelihood function
$\boldsymbol{K}(\boldsymbol{\theta})$	Fisher's information matrix
$\boldsymbol{I}(\boldsymbol{\theta})$	observed information matrix
$\overset{a}{\sim}$	to be asymptotically distributed
$\mathcal{U}$	index sets for uncensored data
$\mathcal{C}$	index sets for censored data
$k(\cdot,\cdot)$	link function
$h_0(\cdot)$	baseline hrf
$S_0(\cdot)$	baseline sf
$\Gamma(\cdot)$	gamma function
$\psi^{(n)}(\cdot)$, $\psi(\cdot)$	polygamma and digamma functions
γ, C	Euler-Mascheroni constant
$\Gamma(\cdot,\cdot)$, $\gamma(\cdot,\cdot)$	incomplete gamma functions
$\Gamma_1(\cdot,\cdot)$, $\gamma_1(\cdot,\cdot)$	regularized incomplete gamma functions
$M(\cdot,\cdot,\cdot)$	Kummer's first kind confluent hypergeometric function
$U(\cdot,\cdot,\cdot)$	Kummer's second kind confluent hypergeometric function
$N(\mu,\sigma^2)$	the normal (Gaussian) distribution with mean μ and variance σ^2
$\Phi(\cdot)$	standard normal cdf
$\phi(\cdot)$	standard normal pdf
$\mathrm{erf}(\cdot)$, $\mathrm{erfc}(\cdot)$	error function amd its counterpart
$B(\cdot,\cdot)$, $B_x(\cdot,\cdot)$	beta and incomplete beta functions
$I_X(\cdot,\cdot)$	beta cdf
$\eta(\cdot)$	Riemann's zeta function
$F(\cdot,\cdot;\cdot,\cdot)$, ${}_2F_1(\cdot,\cdot;\cdot,\cdot)$	confluent hypergeometric function
$(z)_n$	Pochhammer polynomial
$G^{m,n}_{p,q}(\cdot)$	the Meijer G-function
$J_\tau(x)$	Bessel function of the first kind
$F^{(n)}_A(\cdot;\cdot;\cdot;\cdot)$	Lauricella function of type A
$F^{A:B}_{C:D}(\cdot;\cdot;\cdot)$	generalized Kampé de Fériet function
${}_p\Psi_q$	complex parameter Wright generalized hypergeometric function
$(\Omega,\mathcal{F},P)$	probability space
μ'_k, μ_k, $\mu'_{(k)}$	kth moment on zero, kth central moment and descending factorial moment

CV	coefficient of variation
γ_1, γ_2	skewness and kurtosis
$M_X(t)$, $K_X(t)$	mgf and cgf
κ_k	kth cumulant
$\phi_X(t)$	cf
$\tau_{k,l}$	PWM
$B_F(\cdot)$, $L_F(\cdot)$	BC and LC
B, G	BI and GI
$H_{\mathrm{R}}^{\beta}(X)$, $H_{\mathrm{S}}(X)$	Rényi (with order β) and Shannon entropies of X
R	Reliability
$X_{i:n}$, $F_{X_{i:n}}(x)$, $f_{X_{i:n}}(x)$	ith order statistic and its cdf and pdf
$D(\cdot,\cdot)$	distance measure
A^*	modified Anderson-Darling statistic
W^*	modified Cramér-von Mises statistic

Chapter 1

Introduction

Abstract: This chapter presents mathematical and statistical background for understanding this book. Some results and formulae presented in this chapter are revisited in the next chapters. Initially, important survival analysis concepts are defined and issues with respect to inference and statistical methods. Subsequently, several special functions are presented. The chapter ends with some discussions on statistical elements that will be used in the rest of the book.

Keywords:: Censoring data; Inference; Mathematical functions; Statistical functions; Survival functions; Survival regression.

Lifetime statistical analysis is an important subject in applied areas like biomedical science, engineering, reliability, social sciences and several others. Typically, the term *lifetime* or *failure* can have different interpretations. According to Lai (2011) [1], it can represent:

- the human life age [2],
- the time operation of an equipment to fail [3],
- the survival time of a patient with a serious disease from the date of diagnosis [4],
- the time to first recurrence of a tumor (*i.e.*, length of remission) after initial treatment or
- the duration of a social event such as marriage [2].

In above practical occurrences, a failure can not be computed either by an imposed contextual criterion or due to a stochastic censoring. For instance, it

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

can be seen whereas a patient does not die during a clinical treatment period or if he (or she) leaves the trial process. Thus, the proposal of analysis methods that incorporate censoring as well as procedures for failure time data has been sought. *Survival analysis* is the set of statistical procedures able to describe time-to-event censored data. An important step to deal with survival data consists at proposing more flexible models, which furnish a good representation for both nature of data and the shape of its empirical distribution. This book presents a comprehensive mathematical treatment about the main classes of distributions for describing lifetime data.
New distributions often result from a modification of a baseline random variable X by (i) linear transformation, (ii) power transformation (*e.g.* the Weibull is obtained from the exponential), (iii) non-linear transformation (*e.g.* the log-normal from the normal), (iv) log transformation (*e.g.* the log Weibull, also known as the type 1 extreme value distribution), and (v) inverse transformation (*e.g.* the inverse Weibull and inverse gamma models). In what follows, we present two simple transformations for generating new models.

POWER TRANSFORMATION

Consider $G(x)$ be the original cumulative distribution function (cdf) and $F(x)$ be the cdf of a new ageing distribution derived from $Y \sim G$ by exponentiating as follows:

$\boxed{F(t) \,=\, G(t)^{\alpha}}$**:** Using such power transformation, one can deduce the generalized modified Weibull distribution proposed by Carrasco *et al.* (2008) [5], the exponentiated Erlang distribution by Lai (2010) [6] and the exponentiated Weibull by Mudholkar and Srivastava (1993) [7].

$\boxed{F(t) \,=\, 1 \,-\, \{\,1 \,-\, G(t)\,\}^{\beta}}$**:** The Lomax model can be formulated from the Pareto distribution in this way.

MIXTURE OF DISTRIBUTIONS

New models are often obtained from mixtures of two or more distributions. Let π be the mixing proportion of two cdfs $F_1(t)$ and $F_2(t)$. The cdf $F(t)$ resulting from mixture between the two cdfs is

$$\boxed{F(t) \,=\, \pi\, F_1(t) \,+\, (1-\pi)\, F_2(t).}$$

In this book, we present the formalisms of five new classes or generators of distributions, which have been used for describing high complexity data; in particular, for the survival analysis context. The background for understanding class concepts and associated applications is presented in the rest of this chapter.

1.1. PRIMARY DEFINITIONS

Let $T \geq 0$ denote the lifetime random variable having $f_T(t)$ and $F_T(t) = \int_0^t f_T(x)\,\mathrm{d}x$ as probability density function (pdf) and cdf, respectively. In this book, we consider that T is an absolutely continuous random variable (for a discussion on discrete lifetime models, see Lawless (1982, p. 10) [8]). In this case, $S(t) = \bar{F}_T(t) = 1 - F_T(t) = \int_t^\infty f_T(x)\,\mathrm{d}x$ is defined as *reliability* or *survival function* (sf). It is obvious that $S(t)$ is a monotone decreasing function with $S(0) = 1$ and $S(\infty) = \lim_{t\to\infty} S(t) = 0$. The pdf can be expressed in terms of $S(t)$ as

$$f_T(t) = \lim_{\Delta t \to 0^+} \frac{P(t \leq T < t + \Delta t)}{\Delta t} = \frac{\mathrm{d}F_T(t)}{\mathrm{d}t} = -\frac{\mathrm{d}S(t)}{\mathrm{d}t}.$$

An important concept is the *hazard rate function* (hrf) defined as

$$h_T(t) = \lim_{\Delta t \to 0^+} \frac{P(t \leq T < t + \Delta t \mid T \geq t)}{\Delta t} = \frac{f_T(t)}{1 - F_T(t)} = \frac{f_T(t)}{S(t)} \tag{1.1}$$

and, therefore, "$h_T(t)\,\Delta t$" returns the probability of failure in $(t, t + \Delta t]$ given the "unit" has survived until time t. The hrf is also referred to as the *risk* or *mortality rate*. The functions $\bar{F}_T(\cdot)$ and $h_T(\cdot)$ are also called as *ageing* measures. There are several other measures of ageing, but we discuss the hazard and survival functions because they are the major ones in reliability practice. Further, we have

$$h_T(t) = -\frac{\mathrm{d}S(t)/\mathrm{d}t}{S(t)} = -\frac{\mathrm{d}\log[S(t)]}{\mathrm{d}t}$$

and, therefore, the *cumulative hazard function* (chf), $H_T(t)$, is

$$\begin{aligned} H_T(t) &= \int_0^t h_T(u)\,\mathrm{d}u = -\log[S(t)] \\ &\Leftrightarrow S(t) = \exp\left[-H_T(t)\right] = \exp\left[-\int_0^t h_T(u)\,\mathrm{d}u\right]. \end{aligned}$$

Thus, the pdf of T can be expressed from (1.1) as

$$f_T(t) = h_T(t)\,\exp\left[-\int_0^t h_T(u)\,\mathrm{d}u\right].$$

Moreover, from probability basic results for non-negative random variables, one has that $\mathrm{E}(T) = \int_0^\infty S(t)\,\mathrm{d}t$, *i.e.*, the mean survival time is the total

Chapter 2

Exponentiated Models

Abstract: The exponentiation transform of cumulative distributions can furnish more flexible models. Such procedure generates the exponentiated G (exp-G) distributions. This chapter presents a survey on the exp-G models and its mathematical properties. In particular, explicit expressions for the quantile function, ordinary and incomplete moments, generating function, income measures, order statistics and entropies are addressed.

Keywords:: Exponentiated model; Generating function; Hazard function; Moment; Weibull distribution.

2.1. INTRODUCTION

The proposal of more flexible distributions is an activity often required in practical contexts. In particular, adding a positive real parameter (say $\alpha > 0$) to a cdf $G(x)$ by exponentiation gives a cdf $G(x)^\alpha$ that can provide interesting mathematical properties and better fits to data sets in different contexts. Several works have provided evidence that such class covers both monotonic and non-monotonic hazard rates [25, 26]. Despite simplicity of the approached transformation, the resulting distribution is richer than the baseline $G(x)$. Thus, a tailored treatment for this transformation is required.

Let $G(x)$ and $g(x)$ be the cdf and pdf, respectively, of a known random variable (say, a *baseline model*). A random variable X is said to have the *exponentiated-G* ("exp-G" for short) class if its cdf and pdf are

$$F(x) = G(x)^\alpha, \quad \text{for } x \in \mathcal{D} \subset \mathbb{R} \tag{2.1}$$

and

$$f(x) = \alpha\, g(x)\, G(x)^{\alpha-1}, \tag{2.2}$$

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

respectively. We omit the parametric elements for simplicity. This model is denoted by $X \sim$ exp-G($\boldsymbol{\theta}$), where $\boldsymbol{\theta} = (\alpha, \boldsymbol{\delta}^\top)^\top \in \boldsymbol{\Theta} \subset \mathbb{R}^{p+1}$, $^\top$ is the transposition operator, α represents the additional parameter, $\boldsymbol{\Theta}$ is the parametric space of the generated exp-G distribution and $\boldsymbol{\delta}$ is the p-dimensional vector of parameters of the baseline distribution. As one of its properties, the exp-G class can be understood as the *proportional reversed hazard rate model*. In summary, the reversed hazard rate function (rhrf) is the probability of observing an outcome within a neighborhood of x, conditional on the outcome being no more than x, and it is defined (for any baseline model) by $\lambda_G(x) = \mathrm{d}\{\log[G(x)]\}/\mathrm{d}x = g(x)/G(x)$ [27, 28]. Thus, the exp-G rhrf is

$$\lambda_F(x) = \frac{\alpha\, g(x)\, G(x)^{\alpha-1}}{G(x)^{\alpha}} = \alpha\, \lambda_G(x),$$

i.e., the rhrf of the exp-G class is a proportional rhrf. Models which satisfy this characteristic have been sought in the lifetime data analysis literature [29, 30]. An important aspect is that the class determined by (2.1) and (2.2) under $\alpha \in \mathbb{N}$ was pioneered by Lehmann (1953) [31], called initially by *Lehmann alternative type I*. The physical interpretation of the additional parameter of the exp-G class is discussed as follows:

- $F(x) = G(x)$ for $\alpha = 1$;
- For $\alpha = n \in \mathbb{N}$, $F(x)$ represents the cdf of the maximum value defined on a n-variate random sample from $Y \sim G$, say $\{Y_1, \ldots, Y_n\}$:

$$X(n) = \max\{Y_1, \ldots, Y_n\}.$$

The subsequent discussion emphasizes the importance of exp-G distributions. Consider a biological situation, on which corrupted cells are battling to provide observable tumours. Set X_j, $j = 1, \ldots, N$, as the time for the jth corrupted cell to become in a observable tumour (promotion time), where N means the latent number of corrupted cells which may furnish the interest event. Admit N having probability mass function (pmf) given by $p_n = \Pr(N = n)$ for $n = 0, 1, \ldots$ Let $A_N(s) = \sum_{n=0}^{\infty} p_n s^n$ be the corresponding probability generating function (pgf) for $0 < s < 1$, and p_0 the cure rate. Conditional on N, assume that the X_j's are independent random variables having a common baseline pdf $g(x)$ and survival function $S(x) = 1 - G(x)$. Given $N = n$ and the lifetime $T = t$, let Z_j, $j = 1, \ldots, n$, be independent random variables, independently of N, following a Bernoulli distribution with success probability $G(t)$ indicating the presence of the jth competing cause at time t. Further, we consider that the population is divided into two sub-populations of cured and non-cured patients defined by: $U_t = 1$ if $Z_1 + \cdots + Z_N = 1$ and $U_t = 0$ if $Z_1 + \cdots + Z_N = 0$, where $\Pr(U_t = 1) = 1 - p_0$. Let T be a non-negative lifetime random variable and X the promotion time with pdf $g(x)$. Define the distribution of T as the conditional distribution of X, given $U_t = 1$. Under this set up, Rodrigues *et al.* (2011) [32] demonstrated that the pdf of T is given by

$$f_T(t) = \frac{g(t)}{1-p_0} \left\{ \left. \frac{\mathrm{d}A_N(s)}{\mathrm{d}s} \right|_{s=S(t)} \right\}.$$

The corresponding hrf is

$$h_T(t) = \frac{g(t)}{A_N\left(S(t)\right) - p_0} \left\{ \left. \frac{\mathrm{d}A_N(s)}{\mathrm{d}s} \right|_{s=S(t)} \right\}.$$

The class of distributions specified by the pdf $f_T(t)$ is quite broad. It contains as special cases either exp-G distributions or generalizations from them. For example, the generalized exponential Poisson (Barreto-Souza and Cribari-Neto, 2009 [33]), Lehmann alternatives, Weibull-geometric (Barreto-Souza *et al.*, 2010 [34]), exponentiated Weibull (EW) [25], generalized modified Weibull (Carrasco *et al.*, 2008a [5]) and exponential power series (Chahkandi and Ganjali, 2009 [35]) distributions. The properties of the exponentiated distributions were widely discussed in the last years, see Mudholkar and Srivastava (1993) [7] and Mudholkar *et al.* (1995) [36] for exponentiated Weibull, Gupta *et al.* (1998) [37] and Gupta and Kundu (2001) [38] for exponentiated exponential, Nadarajah and Gupta (2007) [39] for exponentiated gamma, Carrasco *et al.* (2008) [5] for exponentiated modified Weibull and Cordeiro *et al.* (2011) [40] for exponentiated generalized gamma distribution.

2.2. SPECIAL CASES

In this section, we discuss some special cases of the exp-G class. As the first case, Gupta *et al.* [37] pioneered the *exponentiated exponential* (EE) distribution as a generalization of the standard exponential distribution. Its two parameters represent the shape and the scale parameters as those cases of the classical gamma and Weibull distributions. The mathematical properties of the EE distribution were studied by Nadarajah and Kotz [41] and also by Gupta and Kundu [42]. Several papers have addressed other properties: see Gupta and Kundu (2001a, 2001b, 2007) [38, 43, 44], Raqab and Ahsanullah (2001) [45], Raqab (2002) [46], Sarhan (2007) [47], Abdel-Hamid and Al-Hussaini (2009) [48], and Aslam *et al.* (2010) [49]. Four special cases of the exponentiated distributions are discussed in Nadarajah and Kotz (2006) [41]:

Chapter 3

Beta Generalized Models

Abstract: The beta transformation gives a great variety of shapes which allow to model symmetric, skewed and bimodal shaped densities. Such procedure generates what we call the beta generalized (beta-G) family of distributions. This chapter presents a survey on the beta-G models and their mathematical properties. We present some explicit expressions for the ordinary and incomplete moments, probability weighted moments, cumulants and generating function, mean deviations, entropies and order statistics.

Keywords:: Beta-G model; Entropy; Mean deviation; Moment; Probability weighted moment; Regression model.

3.1. INTRODUCTION

Several distributions have been employed in order to perform inference on populational properties from one or more observed samples. Choosing the model which should be adopted to test hypothesis about the data is a crucial step in statistical data analysis. In this chapter, the *beta-G* ("BG") family of distributions proposed by Eugene *et al.* (2002) [101] is studied in details.
This family includes nearly all of well-known models as special or limiting cases such as those exponentiated distributions. Further, it can give lighter tails and heavier tails and be applied in some areas such as engineering and biological research, among others. Explicit expressions are reported, which facilitate to obtain several mathematical properties of this family.

In the last years, several BG models have been proposed in this family, mostly by statisticians in Brazil. This family has the major benefit for fitting skewed data that can not be fitted by most well-known continuous distributions. Starting from a baseline cdf $G(x; \boldsymbol{\tau})$, where $\boldsymbol{\tau}$ indicates the parameters of

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

the cdf $G(\cdot)$, Eugene *et al.* (2002) [100] defined the BG family by the cdf (for $x \in \mathbb{R}$)

$$F(x;a,b,\boldsymbol{\tau}) = I_{G(x;\boldsymbol{\tau})}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x;\boldsymbol{\tau})} \omega^{a-1}(1-\omega)^{b-1} d\omega, \qquad (3.1)$$

where $a > 0$ and $b > 0$ are shape parameters, $I_y(a,b) = B_y(a,b)/B(a,b)$ is the incomplete beta function ratio, $B_y(a,b) = \int_0^y \omega^{a-1}(1-\omega)^{b-1} d\omega$ is the incomplete beta function, $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(a) = \int_0^\infty \omega^{a-1}\mathrm{e}^{-\omega}\mathrm{d}\omega$ is the gamma function. If $G(x;\boldsymbol{\tau}) = x$ for $x \in (0,1)$, we obtain the beta distribution. The pdf associated with (3.1) can be written as (for $x \in \mathbb{R}$)

$$f(x;a,b,\boldsymbol{\tau}) = \frac{1}{B(a,b)}\, g(x;\boldsymbol{\tau}) G(x;\boldsymbol{\tau})^{a-1}[1-G(x;\boldsymbol{\tau})]^{b-1}, \qquad (3.2)$$

where $g(x;\boldsymbol{\tau}) = \mathrm{d}G(x;\boldsymbol{\tau})/\mathrm{d}x$ is the baseline pdf. The manageability of $f(x;a,b,\boldsymbol{\tau})$ is linked with the forms of $G(x;\boldsymbol{\tau})$ and $g(x;\boldsymbol{\tau})$. In fact, depending on the complexity of these functions, we can take considerable time and effort to work with the density (3.2) in generality.

If G has the support in $\mathbb{R}^+$, the BG hrf has the form

$$h(x;a,b,\boldsymbol{\tau}) = \frac{g(x;\boldsymbol{\tau})\, G(x;\boldsymbol{\tau})^{a-1}\, [1-G(x;\boldsymbol{\tau})]^{b-1}}{B\,(a,b)\, [1 - I_{G(x;\boldsymbol{\tau})}(a,b)]}.$$

We use $G(x)$ instead of $G(x;\boldsymbol{\tau})$, $g(x)$ instead of $g(x;\boldsymbol{\tau})$, $F(x)$ instead of $F(x;a,b,\boldsymbol{\tau})$, *etc*, to simplify the notation.

Throughout this chapter, the random variable X having density function (3.2) is denoted by $X \sim \mathrm{BG}(a,b,\boldsymbol{\tau})$. It can be expressed by the stochastic representation $X = Q(U) = F^{-1}(U)$, where $U \sim \mathrm{Beta}(a,b)$ and $Q(\cdot)$ is the inverse function of (3.1). Further, we can write (3.1) in terms of the Gaussian hypergeometric function (Gradshteyn and Ryzhik, 2000; Section 9.1 [11]). The properties of this function are well-known. We have

$$F(x) = \frac{G(x)}{a\,B(a,b)}\, {}_2F_1(a, 1-b; a+1; G(x)), \quad x \in \mathbb{R},$$

where

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)} \sum_{j=0}^{\infty} \frac{\Gamma(a+j)\,\Gamma(b+j)}{\Gamma(c+j)}\, \frac{z^j}{j!}.$$

One important special model of the BG family is the exp-G class, discussed in Chapter 2, which arises when $b = 1$ in (3.2). The BG family received great consideration in the last years, after the proposals of Eugene *et al.* (2002) [100] and Jones (2004) [102]. After these seminal works, many extended models were introduced and studied. Gupta and Nadarajah (2004) [103] determined a more general formula for the nth moment of the beta normal (BN) distribution. Razzaghi (2009) [104] adopted the BN distribution for risk assessment and to model dose-response, where the BN properties are discussed and the risk estimates are based upon the asymptotic properties of the MLEs. Recently, Rêgo *et al.* (2012) [105] furnished a better treatment for the BN distribution, derivingseveral of its properties and a detailed discussion on its bimodality. They derived a power series for the qf to obtain computable expressions for the moments, generating function and mean deviations. Further, the BN law has been employed successfully to synthetic aperture radar imagery processing; see Cintra *et al.* (2012) [106]. fgumbelThese authors proposed the beta generalized normal distribution by compounding the beta and generalized normal distributions.

The first distribution of the BG class was the BN model (Eugene *et al.*, 2002 [100]). Denote the standard normal cdf and pdf by $\Phi(\cdot)$ and $\phi(\cdot)$, respectively. Let $X = \Phi^{-1}(U)$, where $U \sim \text{Beta}(a, b)$. Then, X has a standard BN distribution, say $\text{BN}(a, b, 0, 1)$, if its pdf has the form

$$f(x; a, b, 0, 1) = \frac{1}{B(a, b)}\, \phi(x)\, \Phi(x)^{a-1}\, [1 - \Phi(x)]^{b-1}, \quad x \in \mathbb{R}.$$

The skewness and kurtosis of X usually depend on the extra parameters a and b, see Table 3.1. Eugene *et al.* (2002) evaluated the first four cumulants of X with $\mu = 0$ and $\sigma^2 = 1$ for some values of these parameters between 0.05 and 100. The skewness of X is in the interval $(-1, 1)$ and the largest kurtosis value is 4.1825 for $a = 100$ and $b = 0.1$. If $X \sim \text{BN}(a, b, 0, 1)$, then $Y = \sigma X + \mu \sim \text{BN}(a, b, \mu, \sigma)$ has the non-standard BN distribution with parent $\text{N}(\mu, \sigma^2)$.

New distributions in the BG family were investigated in the last ten years. Some of them are now described in the order they were published. The beta Fréchet (BFr) distribution follows from the Fréchet cdf $G(x)$. It was defined by Nadarajah and Gupta (2004) [103], who studied analytically its density and hrf as well as the limit distribution of the order statistics. Nadarajah and Kotz (2004) [41] proposed the beta Gumbel distribution from the Gumbel cdf $G(x)$ and yielded expressions for its moments and the asymptotic distribution of the order statistics. A generalization of the Weibull model called the beta Weibull (BW) distribution was presented by Famoye *et al.* (2005) [80].

Gupta and Nadarajah (2006) [106] introduced the beta Bessel distribution

Chapter 4

Special Generalized Beta Models

Abstract: In this chapter, we provide theoretical essays about five special models in the beta family. For each model, its cdf, pdf and hrf have explicit forms, which can be utilized for determining some mathematical properties. Two applications are performed in order to illustrate the flexibility of the densities under discussion.

Keywords:: BBS; Beta-G model; BF; BGE; BMW; BW; Moment.

This chapter includes a discussion about five special beta models: beta generalized exponential, beta Weibull, beta Fréchet, beta modified Weibull and beta Birnbaum-Saunders distributions. Two applications to real data with positive support emphasize the flexibility of these five models.

4.1. BETA GENERALIZED EXPONENTIAL

As mentioned in Chapter 2, a random variable T is said to have the generalized exponential (GE) distribution if its cdf and pdf are

$$G(x;\lambda,\alpha) = (1-\mathrm{e}^{-\lambda x})^{\alpha} \quad \text{and} \quad g(x;\lambda,\alpha) = \alpha\,\lambda\,\mathrm{e}^{-\lambda\,x}(1-\mathrm{e}^{-\lambda\,x})^{\alpha-1}, \tag{4.1}$$

respectively. The shape ($\alpha > 0$) and scale ($\lambda > 0$) parameters of the GE distribution are similar to those of the gamma and Weibull distributions.

The distribution (4.1) is also named the exponentiated exponential (EE) distribution. Note that the exponential distribution is a special case of the GE

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

distribution when $\alpha = 1$.

The four-parameter beta generalized exponential (BGE) distribution is defined by taking $G(x)$ in equation (4.1) as the baseline cdf in (3.1). Thus, a random variable X is said to have the BGE distribution if its cdf and pdf are (for $x > 0$)

$$F(x; a, b, \lambda, \alpha) = I_{(1-\mathrm{e}^{-\lambda x})^{\alpha}}(a, b) \tag{4.2}$$

and

$$f(x; a, b, \lambda, \alpha) = \frac{\alpha\lambda}{B(a, b)} \mathrm{e}^{-\lambda x}(1 - \mathrm{e}^{-\lambda x})^{\alpha a-1} \left\{1 - (1 - \mathrm{e}^{-\lambda x})^{\alpha}\right\}^{b-1}, \tag{4.3}$$

respectively, for $a, b, \lambda, \alpha > 0$. The corresponding hrf becomes

$$\tau(x; \lambda, \alpha) = \frac{\alpha\lambda \mathrm{e}^{-\lambda x}(1 - \mathrm{e}^{-\lambda x})^{\alpha a-1} \left\{1 - (1 - \mathrm{e}^{-\lambda x})^{\alpha}\right\}^{b-1}}{B(a, b)[1 - I_{(1-\mathrm{e}^{-\lambda x})^{\alpha}}(a, b)]}. \tag{4.4}$$

Note that the pdf (4.3) does not involve any complicated function. If X is a random variable with pdf (4.3), we write $X \sim \mathrm{BGE}(a, b, \lambda, \alpha)$. The BGE distribution generalizes some well-known distributions in the literature. The GE distribution is a special case when $a = b = 1$. For $\alpha = 1$, we obtain the exponential distribution with parameter λ. The BE distribution follows from (4.2) when $\alpha = 1$. The hrf (4.4) can be bathtub shaped, monotonically increasing or decreasing and upside-down bathtub depending on the parameter selection.

There are two simple formulae for the cdf of the BGE distribution depending if the parameter $b > 0$ is real non-integer or integer. Note that, if $|z| < 1$ and $b > 0$ is real non-integer, we have

$$(1 - z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b - j) j!} z^j, \tag{4.5}$$

where $\Gamma(\cdot)$ is the gamma function. Using the expansion (4.5) in (4.2), the cdf of the BGE distribution when $b > 0$ is real non-integer follows as

$$\begin{aligned} F(x; a, b, \alpha, \lambda) &= \frac{\Gamma(b)}{B(a, b)} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(b - j) j!} \int_0^{(1-\mathrm{e}^{-\lambda x})^{\alpha}} \omega^{a+j-1} \mathrm{d}\omega \\ &= \frac{\Gamma(a + b)}{\Gamma(a)} \sum_{j=0}^{\infty} \frac{(-1)^j G(x; \lambda, \alpha\,(a + j))^{\alpha\,(a+j)}}{\Gamma(b - j) j!(a + j)} \\ &= \sum_{j=0}^{\infty} w_j\, G(x; \lambda, \alpha\,(a + j))^{\alpha\,(a+j)}, \end{aligned} \tag{4.6}$$

where $w_j = (-1)^j \Gamma(a+b)[(a+j)\,\Gamma(a)\,\Gamma(b-j)\,j!]^{-1}$. Equation (4.6) reveals that the cdf of the BGE distribution can be expressed as an infinite weighted sum of cdfs of GE distributions. The BE cdf follows with $\alpha = 1$ from (4.6). The cdf of the double generalized exponential (DGE) distribution follows from (4.6) when $a = 1$.

By differentiating (4.6), the density function (4.3) can be expressed as a linear combination of the GE pdfs

$$f(x; a, b, \lambda, \alpha) = \sum_{j=0}^{\infty} w_j\, f^*(x; \lambda, \alpha\,(a+j)),$$

where $f^*(x; \lambda, \alpha\,(a+j)) = \alpha(a+j)\, g(x; \lambda, \alpha\,(a+j))\, G(x; \lambda, \alpha\,(a+j))^{\alpha(a+j)-1}$ is the GE density function corresponding to the cdf $G(x; \lambda, \alpha\,(a+j))^{\alpha(a+j)}$ in equation (4.6). Thus, the BGE distribution has the advantage that some of its mathematical properties can be directly obtained from the corresponding properties of the GE distribution, such as those explored in Sections 3.5 and 3.8.

4.2. BETA WEIBULL

A random variable T is said to have the Weibull distribution if its cdf and pdf are

$$G(x; \alpha, \beta) = 1 - \mathrm{e}^{-(\beta x)^{\alpha}} \quad \text{and} \quad g(x; \alpha, \beta) = \alpha\, \beta^{\alpha} x^{\alpha-1} \mathrm{e}^{-(\beta x)^{\alpha}}, \quad x > 0,$$

respectively. The four-parameter beta Weibull (BW) cdf is defined by inserting the above $G(x; \alpha, \beta)$ in equation (3.1). Thus, a random variable X is said to have the BW distribution if its cdf and pdf are (for $x > 0$)

$$F(x; a, b, \alpha, \beta) = I_{1-\exp\{-(\beta x)^{\alpha}\}}(a, b) \tag{4.7}$$

and

$$f(x; a, b, \alpha, \beta) = \frac{\alpha\, \beta^{\alpha}}{B(a, b)} x^{\alpha-1} \exp\{-b\,(\beta x)^{\alpha}\}\, [1 - \exp\{-(\beta x)^{\alpha}\}]^{a-1}, \tag{4.8}$$

respectively, for $a > 0$, $b > 0$, $\beta > 0$ and $\alpha > 0$. The associated hrf is

$$\tau(x; a, b, \alpha, \beta) = \frac{\alpha\, \beta^{\alpha} \exp\{-b\,(\beta x)^{\alpha}\}\, [1 - \exp\{-(\beta x)^{\alpha}\}]^{a-1}}{B(a, b) I_{1-\exp\{-(\beta x)^{\alpha}\}}(a, b)}.$$

If X is a random variable with pdf (4.8), we write $X \sim BW(a, b, \alpha, \beta)$. The BW distribution contains as special case the EW distribution when $b = 1$.

Chapter 5

The Kumaraswamy's Generalized Family of Models

Abstract:
This chapter addresses the Kumaraswamy's generalized ("Kw-G" for short) family of models. A physical motivation for the Kw-G family is presented and some of its special cases are discussed in detail. This family receives a baseline distribution as input and returns a new distribution with two additional parameters. The returned model is often more flexible than the baseline one. Several structural properties are presented and discussed for the Kw-G family. Among them, useful expansions, mgf, moments and mean deviations. Additionally, estimation and generation procedures are presented.

Keywords:: Asymptotes; Kw-G Model; Moment; Physical Motivation; Shapes.

5.1. INTRODUCTION

In life testing experiments, the data can be modeled by a wide range of distributions. Kumaraswamy (1980) [162] pioneered a distribution for double bounded random processes with applications in hydrology. In addition to the hydrological context, the Kumaraswamy (Kw) model has been adopted in related areas, such as reservoir operations and design, see, for example, Fletcher and Ponnambalam (1996) [163] and Seifi *et al.* (2000) [164].
The pdf and cdf of the Kw distribution with two shape parameters $a > 0$ and $b > 0$ in the interval $(0, 1)$ are, respectively,

$$\pi(x) = a\,b\,x^{a-1}(1-x^a)^{b-1} \quad \text{and} \quad \Pi(x) = 1-(1-x^a)^b. \tag{5.1}$$

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

The density function (5.1) has several properties similar to those of the beta distribution but has some advantages in terms of tractability.
The Kw distribution is not widely known, although Jones (2009) [164] pointed out some differences and similarities with the beta distribution. For example, the Kw densities can be unimodal, anti-modal, increasing, decreasing or constant depending on the parameter values in a similar way of the beta distribution. He addressed several advantages of this distribution over the beta distribution such as simpler formulae for the cdf and qf and moments of the order statistics. Jones (2009) [164] also emphasized that the beta distribution has some advantages over the Kw distribution such as simpler expressions for moments and generating function and more ways for generation using physical processes.
The Kumaraswamy generalized ("Kw-G") family of distributions was proposed by Cordeiro and de Castro (2011) [87] and has the Kw distribution as the baseline model. The Kw-G family is defined as

$$F(x) = 1 - \{1 - G(x)^a\}^b, \tag{5.2}$$

Note that the two additional parameters $a > 0$ and $b > 0$ provide skewness and vary tail weights. Because of the simple form of equation (5.2), this family can be easily fitted even if the data are censored. The Kw-G family allows for greater flexibility of its tails and can be applied in several areas of engineering, medicine and biology.
Correspondingly, the pdf and hrf of this family have very simple forms:

$$f(x) = a\, b\, g(x)\, G(x)^{a-1}\, \{1 - G(x)^a\}^{b-1}. \tag{5.3}$$

and

$$\tau(x) = a\, b\, g(x)\, G(x)^{a-1} \{1 - G(x)^a\}^{-1}. \tag{5.4}$$

In this chapter, X denotes the random variable with density function (5.3) and we write $X \sim$Kw-G(a, b). Each Kw-G generated distribution can be determined from a parent G distribution, which is clearly a basic exemplar of the Kw-G family when $a = b = 1$.
If $b = 1$, we obtain as a special case from (5.3) the exp-G family discussed in Chapter 2. One major benefit of the Kw-G family is to fit skewed data that can not be fitted by classic distributions. Most of the results of this chapter follow Cordeiro and de Castro (2011) [87] and Nadarajah *et al.* (2012) [88].
Based on the cdf $G(x)$ and pdf $g(x)$ of any baseline G distribution, we can associate the Kw-G density (5.3) with two extra shape parameters a and b. These parameters can generate distributions with heavier or lighter tails and control skewness and kurtosis. They can provide more flexible distributions. The Kw-G family has a wide variety of shapes and it is able to model bathtub-shaped hazard rate data. Further, it can be easily used for discriminating

between the Kw-G and G distributions. If $a < 1$, then the tails of $f(x)$ will be heavier than those of $g(x)$. Similarly, if $b < 1$, then the tails of $f(x)$ will be heavier than those of $g(x)$. On the other hand, if $a > 1$, then the tails of $f(x)$ will be lighter than those of $g(x)$. Similarly, if $b > 1$, then the tails of $f(x)$ will be lighter than those of $g(x)$. The density (5.3) has an important advantage over the BG class (Eugene *et al.*, 2002 [100]) discussed in Chapter 3, since it does not involve any complicated function.

Each Kw-G distribution can be determined from a given G distribution as follows: the Kw-normal (KwN) distribution follows by taking $G(x)$ in equation (5.3) as the normal cdf. In a similar manner, the Kw-gamma (KwGa), Kw-Weibull (KwW) and Kw-Gumbel (KwGu) models follow by taking $G(x)$ to be the cdf of the gamma, Weibull and Gumbel distributions, respectively.

In this chapter, equation (5.3) is applied in some generality. The structural properties of the Kw-G family are usually simpler to derive than those of the BG family.

If $g(x)$ is a symmetric function around zero, then $f(x)$ will not be a symmetric distribution even when $a = b$. By inverting (5.2), the Kw-G qf can be expressed in terms of the baseline qf, say $Q_G(u) = G^{-1}(u)$, by $Q(u) = Q_{\mathrm{G}}(\{1-(1-u)^{1/b}\}^{1/a})$.

5.2. PHYSICAL MOTIVATION

For a and b positive integers, a physical interpretation of the Kw-G family (5.3) can be given as follows. Suppose that a system is composed by b independent components, which in turn is composed by a independent subcomponents. Define $X_{j1}, \ldots, X_{ja}$ as the subcomponents lifetimes of the jth component (for $j = 1, \ldots, b$) with common cdf $G(x)$. Suppose that the system failure occurs if any of the b components fails and each component fails only if all a subcomponents fail. Further, denote X_j as the lifetime of the jth component (for $j = 1, \ldots, b$) and let X denote the lifetime of the entire system. Then, the cdf of X can be expressed as

$$\begin{aligned}\Pr(X \leq x) &= 1 - \Pr\left(X_1 > x, \ldots, X_b > x\right) = 1 - \Pr^b\left(X_1 > x\right)\\ &= 1 - \left\{1 - \Pr\left(X_1 \leq x\right)\right\}^b = 1 - \left\{1 - \Pr\left(X_{11} \leq x, \ldots, X_{1a} \leq x\right)\right\}^b\\ &= 1 - \left\{1 - G^a(x)\right\}^b.\end{aligned}$$

Hence, the Kw-G family (5.2) is precisely the time to failure distribution of the entire system.

Chapter 6

Special Kumaraswamy Generalized Models

Abstract: In this chapter, three special cases of the Kumaraswamy generalized family of distributions are presented. Some mathematical characteristics are provided such as the moments and generating function. An expanded expression for the density function and some special cases are presented. For illustrative purposes, practical examples of the KwG models are reported by means of applications to empirical data.

Keywords: Kw-G model; KwBXII; KwGum; KwW; Moment.

The cdf and pdf of the Kumaraswamy-G (Kw-G) family (Cordeiro and de Castro, 2010 [87]) are given by

$$F(x) \,=\, 1 \,-\, \{1 - G(x)^a\}^b \tag{6.1}$$

and

$$f(x) = a\, b\, g(x)\, G(x)^{a-1}\, \{1 - G(x)^a\}^{b-1}, \tag{6.2}$$

respectively, for $a > 0$ and $b > 0$, where $G(x)$ and $g(x)$ are the cdf and pdf of an arbitrary baseline distribution, respectively. We denote by $X \sim$Kw-G(a, b) a random variable with cdf (6.1) and pdf (6.2).

Next, we present some mathematical characteristics of three important special models of the Kw-G family, namely: the Kumaraswamy Weibull (Cordeiro *et al.*, 2010 [84]), Kumaraswamy Burr XII (Paranaíba *et al.*, 2011 [117] and 2013 [169]) and Kumaraswamy Gumbel (Cordeiro *et al.*, 2012d [170]) distributions.

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

6.1. KUMARASWAMY WEIBULL

The Weibull distribution has been widely used along several decades in a variety of research areas, such as engineering, medicine and biological sciences, among others. In the last two decades, several extensions of the Weibull distribution were proposed. In this sense, we have the exponentiated Weibull (EW) (Mudholkar *et al.*, 1995 [36], Mudholkar and Hutson 1996 [59]), additive Weibull (Xie and Lai, 1995 [171]), extended Weibull (Xie *et al.*, 2002 [156]), modified Weibull (MW) (Lai *et al.*, 2003 [72]), beta exponential (BE) (Nadarajah and Kotz, 2005 [41]), beta Weibull (BW) (Lee *et al.*, 2007 [81]), extended flexible Weibull (Bebbington *et al.*, 2007 [166]), generalized modified Weibull (GMW) (Carrasco *et al.*, 2008 [5]) and generalized inverse Weibull (Gusmão *et al.*, 2009 [140]) distributions.

In this section, we address some mathematical properties of the KwW model in order to increase the range of possible applications. Note that the Weibull distribution is a basic exemplar of the KwW distribution. Most of the KwW properties presented here were derived by Cordeiro *et al.* (2010a) [84].

The cdf and pdf of the KwW distribution are obtained from (6.1) and (6.2) by taking $G_{\alpha,\beta}(x) = 1 - \exp\{-(\beta x)^\alpha\}$, *i.e.*, the Weibull cdf with parameters $\alpha > 0$ and $\beta > 0$. Hence, we obtain

$$F(x) = 1 - \{1 - [1 - \exp\{-(\beta x)^\alpha\}]^a\}^b \tag{6.3}$$

and

$$\begin{aligned} f(x) &= a\, b\, \alpha\, \beta^\alpha\, x^{\alpha-1} \exp\{-(\beta x)^\alpha\}[1 - \exp\{-(\beta x)^\alpha\}]^{a-1} \\ &\quad \times \{1 - [1 - \exp\{-(\beta x)^\alpha\}]^a\}^{b-1}\,. \end{aligned} \tag{6.4}$$

Hereafter, the random variable X following (6.4) with parameters a, b, α and β is denoted by $X \sim \mathrm{KwW}(a, b, \alpha, \beta)$.

It is clear that the Weibull, EW and EE models are special cases of (6.4) when $a = b = 1$, $b = 1$, and $\alpha = b = 1$, respectively. The KwW distribution (6.4) is much more flexible than its special cases.

The hrf of X is

$$\tau(x) = \frac{ab\alpha\, \beta^\alpha\, x^{\alpha-1} \exp\{-(\beta x)^\alpha\}\, [1 - \exp\{-(\beta x)^\alpha\}]^{a-1}}{1 - [1 - \exp\{-(\beta x)^\alpha\}]^a}. \tag{6.5}$$

Further, the asymptotes of $f(x)$ and $F(x)$ as $x \to 0, \infty$ are given by

$$f(x) \sim a\, b\, \alpha \beta^{a\alpha} x^{a\alpha-1} \quad \text{as} \quad x \to 0,$$

$$f(x) \sim a^b b\, \alpha\, \beta^\alpha\, x^{\alpha-1} \exp\{-b(\beta x)^\alpha\} \quad \text{as} \quad x \to \infty,$$

$$F(x) \sim b(\beta x)^{a\alpha} \quad \text{as} \quad x \to 0,$$

$$1 - F(x) \sim a^b \exp\{-b(\beta x)^\alpha\} \quad \text{as} \quad x \to \infty.$$

Note that the upper tail of $f(x)$ is exponential and the lower tail is polynomial.

6.1.1. Linear Representation

Expanding the binomial $\{1 - G^a(x)\}^{b-1}$ in equation (6.2), the Kw-G family density can be expressed as

$$f(x) = \sum_{j=0}^{\infty} \frac{(-1)^j\, b}{(j+1)} \binom{b-1}{j} h_{(j+1)a}(x), \tag{6.6}$$

where $h_a(x) = ag(x)\, G(x)^{a-1}$ represents the exp-G density with parameter $a > 0$ (Eugene *et al.*, 2002 [100]) (see Section 2.1). In Chapter 2, we obtain some mathematical properties of the exponentiated models.
The KwW density can be expressed as a linear combination of Weibull densities by applying (6.6) to the Weibull distribution and expanding the generalized binomial. We obtain

$$f(x) = \sum_{k=0}^{\infty} w_k\, g_{\alpha,\beta_k}(x), \tag{6.7}$$

where $g_{\alpha,\beta_k}(x)$ is the Weibull density function with parameters α and $\beta_k = \beta\,(k+1)^{1/\alpha}$, and the coefficients w_k are given by

$$w_k = \sum_{j=0}^{\infty} \frac{a\, b\, (-1)^{j+k}}{(k+1)} \binom{b-1}{j} \binom{(j+1)\, a - 1}{k}. \tag{6.8}$$

It is easily verified using `Maple` that $\sum_{k=0}^{\infty} w_k = 1$ as expected. By integrating (6.7), we have

$$F(x) = \sum_{k=0}^{\infty} w_k\, G_{\alpha,\beta_k}(x). \tag{6.9}$$

6.1.2. Moments

Based on equation (6.7), some structural properties like ordinary, incomplete, factorial and inverse moments of X can be determined as infinite linear combinations of the corresponding Weibull quantities. For example, the sth ordinary

Chapter 7

The Gamma-G Family of Distributions

Abstract: This chapter presents the gamma generalized family of distributions proposed by Zofragos and Balakrishnan (2009). Several mathematical properties are provided such as representations for gamma-G density and cumulative functions, some generalized moments, quantile and generating functions and entropies. A bivariate generalization is presented. An application is performed in order to illustrate empirically the usefulness of this family.

Keywords:: Gamma-G Model; GGum; GLL; GLN; GN; GW; Mean deviation; Moment; Order statistic.

7.1. INTRODUCTION

Zografos and Balakrishnan (2009) [202] pioneered a family of univariate continuous distributions generated by gamma random variables. Let $G(x)$ be any parent cdf for $x \in \mathbb{R}$. They defined the *gamma-G* family with pdf $f(x)$ and cdf $F(x)$ given by

$$f(x) = \frac{1}{\Gamma(a)}\, \{-\log[1-G(x)]\}^{a-1}\, g(x) \tag{7.1}$$

and

$$F(x) = \frac{\gamma\left(a, -\log\left[1-G(x)\right]\right)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{-\log[1-G(x)]} t^{a-1}\, \mathrm{e}^{-t} \mathrm{d}t, \tag{7.2}$$

respectively, for $a > 0$, where $g(x) = \mathrm{d}\, G(x)/\mathrm{d}\, x$, $\Gamma(a)$ is the gamma function, and $\gamma(a, z)$ is the incomplete gamma function defined by (1.7) in Section 1.4.

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

The hrf corresponding to (7.1) becomes

$$h(x) = \frac{\{-\log[1-G(x)]\}^{a-1}\, g(x)}{\Gamma\left(a, -\log\left[1-G(x)\right]\right)}, \tag{7.3}$$

where $\Gamma(a,z) = \int_z^\infty t^{a-1}\,\mathrm{e}^{-t}\mathrm{d}t$ denotes the upper incomplete gamma function.

The gamma-G family has the same parameters of the parent G plus an extra shape parameter $a > 0$. Henceforth, if X is a random variable with pdf (7.1), we write $X \sim$ gamma-G(a). Every new gamma-G model can be determined from a given G distribution. Clearly, the G distribution is the basic exemplar of the gamma-G family when $a = 1$.

Zografos and Balakrishnan (2009) [201] presented several motivations for the gamma-G family: if $X_{1:1}, \ldots, X_{1:n}$ are the order statistics from a sequence of independent random variables with common pdf $g(\cdot)$, then the pdf of the nth lower statistic is given by (7.1). Further, if Z is a gamma random variable with shape parameter $a > 0$ and unit scale parameter, then $X = F^{-1}(\exp(Z))$ has the pdf (7.1). Finally, if Z is a log-gamma random variable, then $X = F^{-1}(\exp\{-\exp(Z)\})$ has the pdf (7.1).

Recently, several mathematical properties of (7.1) and (7.2) were investigated by Nadarajah *et al.* (2015) [202]. Zografos and Balakrishnan (2009) [201] proposed expressions for moments associated with special gamma-G models (which hold only for natural a), a general expression for the Shannon entropy and a maximum entropy characterization.

7.2. SPECIAL GAMMA-G MODELS

The gamma-G family density function (7.1) furnishes for greater flexibility to describe tail points and, therefore, can be widely employed in many areas of engineering and biology. In this section, we present five special cases of this family. Models deduced from the Equation (7.1) can be analytically tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions.

7.2.1. The Gamma-Weibull Distribution

Consider $G(x) = 1 - \exp\{-(\beta x)^\alpha\}$ to be the Weibull cdf with scale parameter $\beta > 0$ and shape parameter $\alpha > 0$, the gamma-Weibull (GW) density function (for $x > 0$) becomes

$$f_{\mathcal{GW}}(x) = \frac{\alpha\,\beta^{\alpha a}}{\Gamma(a)}\, x^{a\alpha-1}\, \exp\{-(\beta x)^\alpha\}. \tag{7.4}$$

Equation (7.4) is important because it extends many distributions previously considered in the literature. In fact, it is identical to the generalized gamma (Stacy, 1962 [203]) distribution.

The Weibull distribution is a special case when $a = 1$ and the gamma distribution is another special case when $\alpha = 1$. The half-normal distribution corresponds to $a = 3$ and $\alpha = 2$. In addition, the log-normal distribution is a limiting special case when a goes to infinity.

The cdf and hrf corresponding to (7.4) are

$$F_{\mathcal{GW}}(x) = \frac{\gamma[a, (\beta\, x)^{\alpha}]}{\Gamma(a)}$$

and

$$h_{\mathcal{GW}}(x) \;=\; \frac{\alpha\, \beta^{a\,\alpha}\, x^{a\alpha-1}\, \exp\{-(\beta x)^{\alpha}\}}{\Big\{\Gamma(a) \;-\; \gamma[a, (\beta\, x)^{\alpha}]\Big\}},$$

respectively.

7.2.2. The Gamma-Normal Distribution

The gamma-normal (GN) distribution is defined from (7.1) by taking $G(x)$ and $g(x)$ to be the cdf and pdf of the normal $N(\mu, \sigma^2)$ distribution. Its pdf is

$$f_{\mathcal{GN}}(x) = \frac{1}{\Gamma(a)}\Big\{ -\log\Big[1-\Phi\Big(\frac{x-\mu}{\sigma}\Big)\Big]\Big\}^{a-1}\phi\Big(\frac{x-\mu}{\sigma}\Big),$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$ is a location parameter, $\sigma > 0$ is a scale parameter, $a > 0$ is a shape parameter, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. For $\mu = 0$ and $\sigma = 1$, we obtain the standard GN distribution. Further, this distribution with $a = 1$ becomes the normal distribution.

7.2.3. The Gamma-Gumbel Distribution

Consider the Gumbel distribution with location parameter $\mu \in \mathbb{R}$ and scale parameter $\sigma > 0$, where the pdf and cdf (for $x \in \mathbb{R}$) are

$$g(x) = \frac{1}{\sigma}\exp\left\{\left(\frac{x-\mu}{\sigma}\right) - \exp\left(\frac{x-\mu}{\sigma}\right)\right\}$$

and

$$G(x) = 1 - \exp\left\{-\exp\left(\frac{x-\mu}{\sigma}\right)\right\},$$

Chapter 8

Recent Compounding Models

Abstract: In this chapter, we introduce a family of models defined by compounding two (a continuous and other discrete) distributions. The new family has as limiting case the adopted baseline distribution. The generated models are frequently more flexible than the baseline distributions. Several mathematical properties such as moments, quantile and generating functions, among others, are provided. Further, the estimation procedure is approched by maximum likelihood. The potentiality of the family of models is illustrated by means of two applications to real data.

Keywords: BSPS; BXIIPS; Compounding Models; EWPS; Generating function; Moment; Order statistic; WPS.

8.1. INTRODUCTION

In this chapter, we review some recent compounding lifetime distributions, which were pioneered by Marshall and Olkin (1998) [4] and after extended by some authors. Several well-known lifetime models, such as the exponential, gamma and Weibull distributions, have been extended by compounding lifetime distributions recently introduced in the statistical literature. The class of compounding distributions allows for the use in industrial applications and biological research. It arises by mixing the power series and lifetime distributions. It is specially useful in a situation "where the failure occurs due to the presence of an unknown number, say N, of initial defects of the same kind and the T's represent their lifetimes and each defect can be detected only after causing failure, in which case it is repaired perfectly" (Adamidis and Loukas, 1998 [208]).

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento

Given N, let $T_1, \ldots, T_N$ be iid random variables having a baseline cdf $G(t; \boldsymbol{\tau})$, where $\boldsymbol{\tau}$ is a vector of parameters and N is a discrete random variable following a zero truncated power series (PS) distribution with probability mass function (pmf) expressed by

$$p_n = P(N = n) = \frac{a_n\, \theta^n}{C(\theta)},\; n = 1, 2, \ldots. \tag{8.1}$$

Note that the coefficient a_n is a function of n and $C(\theta) = \sum_{n=1}^{\infty} a_n\, \theta^n$, with $\theta > 0$ such that $C(\theta)$ is finite. It is important to emphasize that the probability distributions of the form (8.1) have been considered in Boehme and Powell (1968) [208] and Ostrovska (2007) [209]. Table 8.1 lists some PS distributions defined by (8.1) such as the Poisson, logarithmic, geometric and binomial distributions.

Define $X = \min\{T_i\}_{i=1}^{N}$. The conditional cumulative distribution of $X|N = n$ is given by

$$F_{X|N=n}(x) = 1 - [1 - G(x; \boldsymbol{\tau})]^n$$

and then

$$P(X \le x, N = n) = \frac{a_n \theta^n}{C(\theta)} \left\{1 - [1 - G(x; \boldsymbol{\tau})]^n\right\}, \quad x > 0, \quad n = 1, 2, \ldots.$$

Therefore, the marginal cdf of X becomes

$$F(x; \theta, \boldsymbol{\tau}) = 1 - \frac{1}{C(\theta)} C\left\{\theta\left[1 - G(x; \boldsymbol{\tau})\right]\right\}, \quad x > 0. \tag{8.2}$$

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C(\theta)^{-1}$	Θ
Poisson	$n!^{-1}$	$\mathrm{e}^{\theta} - 1$	e^{θ}	e^{θ}	$\log(\theta + 1)$	$\theta \in (0, \infty)$
Logarithmic	n^{-1}	$-\log(1 - \theta)$	$(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$1 - \mathrm{e}^{-\theta}$	$\theta \in (0, 1)$
Geometric	1	$\theta(1 - \theta)^{-1}$	$(1 - \theta)^{-2}$	$2(1 - \theta)^{-3}$	$\theta(\theta + 1)^{-1}$	$\theta \in (0, 1)$
Binomial	$\binom{m}{n}$	$(\theta + 1)^m - 1$	$m(\theta + 1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta - 1)^{1/m} - 1$	$\theta \in (0, 1)$

Table 8.1: Functional quantities for some PS distributions.

The pdf associated to (8.2) is

$$f(x; \theta, \boldsymbol{\tau}) = \frac{\theta}{C(\theta)} g(x; \boldsymbol{\tau})\, C'\left\{\theta\left[1 - G(x; \boldsymbol{\tau})\right]\right\}, \quad x > 0, \tag{8.3}$$

where $g(x;\boldsymbol{\tau})$ is the baseline density function and $C'(\cdot)$ is the first derivative with respect to θ.

The random variable X with density (8.3) is called the *G-power series* (GPS) family and denoted by $X \sim GPS(\theta,\boldsymbol{\tau})$, which is customary for such a name given to the distributions arising by means of the operation of compounding.

Remark. In an analogous way for X, define $Y = \max\{T_i\}_{i=1}^{N}$, where $N \sim PS(\theta)$. Then, the cumulative and density functions of Y are

$$F(y;\theta,\boldsymbol{\tau}) = \frac{1}{C(\theta)} C\left[\theta\, G(y;\boldsymbol{\tau})\right], \quad y > 0$$

and

$$f(y;\theta,\boldsymbol{\tau}) = \frac{\theta}{C(\theta)} g(y;\boldsymbol{\tau})\, C'\left[\theta\, G(y;\boldsymbol{\tau})\right], \tag{8.4}$$

respectively. The family with cumulative distribution (8.4) is a complement of the GPS family and thus, hereafter, it is called the *complementary G-power series* (CGPS) family, denoted by $Y \sim \text{CGPS}(\theta,\boldsymbol{\tau})$.

This type of compounding family is suitable for complementary risks scenarios, where the lifetime corresponding to a particular risk is not perceptible, rather we observe only the maximum lifetime value among all risks. Note that equations (8.3) and (8.4) will be most manageable when both functions $G(x;\boldsymbol{\tau})$ and $g(x;\boldsymbol{\tau})$ have uncomplicated expressions. In general, except for some special choices of these functions, these densities will be difficult to deal with. A positive point of the compounding distributions is that the baseline distribution G is a basic exemplar of the generated model. In addition, the compounding distributions have various interesting applications based on the stochastic representations (8.3) and (8.4), which make them of recognizable scientific relevance from other lifetime distributions. We list below some of these interesting applications.

- *Time to the first failure* (Adamidis and Loukas, 1998 [208] and Kus, 2007 [211]). Consider that a component or system can fail after the ocurrence of a number N of early defects of the same kind, only detected after causing failure and perfectly repaired. If we denote by T_i the time to the device failure due to the ith defect, then the model defined in (8.3) is adequate for modelling the time to the first failure, under the assumptions that the T_i's are iid random variables independent of N, which is defined in (8.1).

- *Reliability.* From the stochastic quantities (8.3) and (8.4), we have that the compounding models can emerge in series and parallel systems with identical components, which appear in many industrial applications and biological organisms.

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SUBJECT INDEX

Gauss M. Cordeiro, Rodrigo B. Silva & Abraão D. C. Nascimento (Eds.)

C

F

G

H

I

K

L

M

N

O

P

Q

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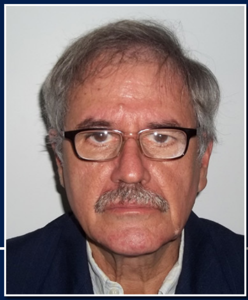

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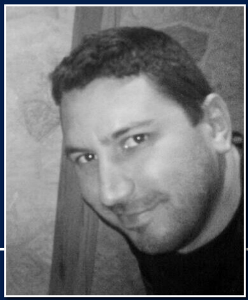

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