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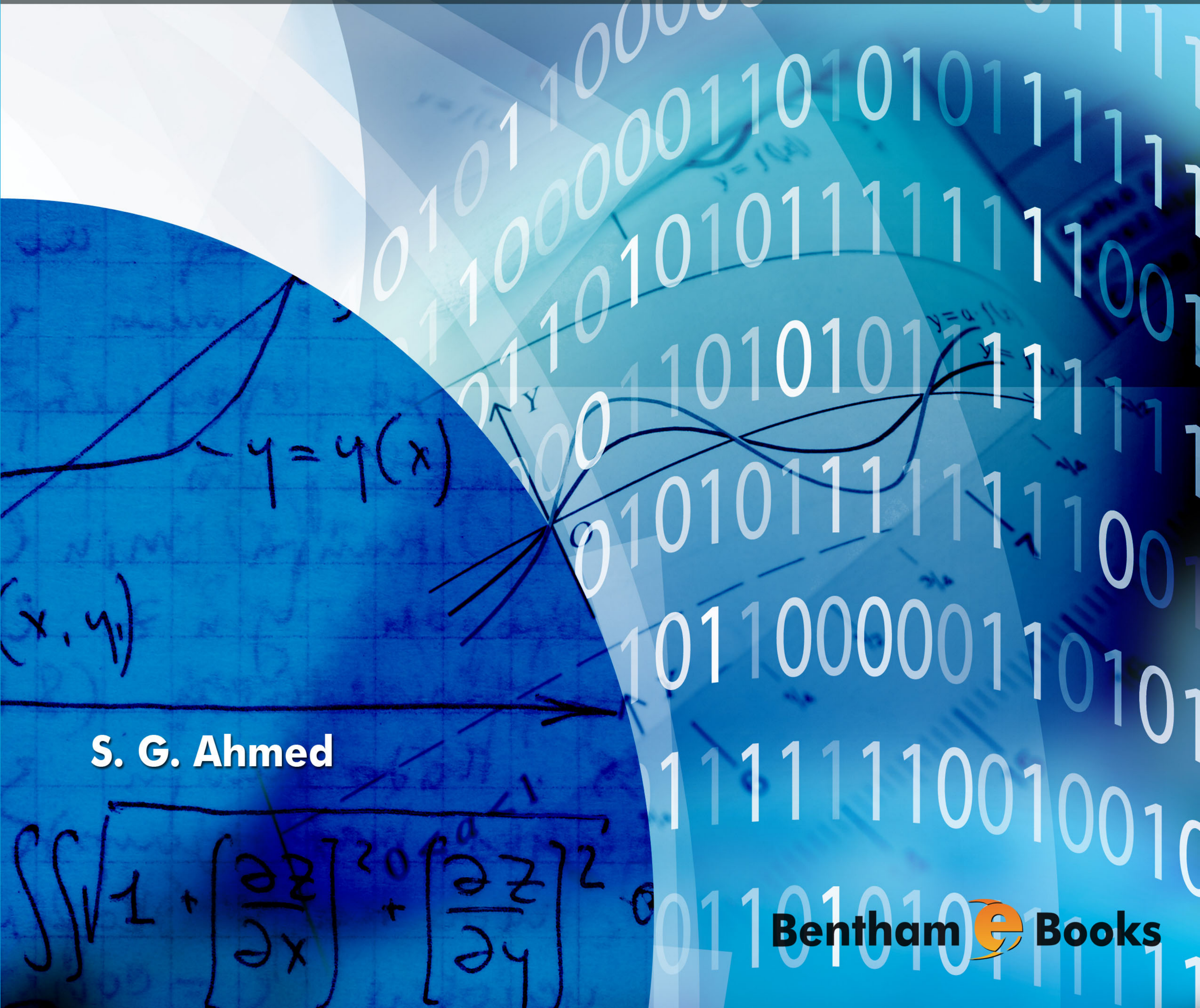
Higher Mathematics for Science, Technology and Engineering



S. G. Ahmed

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Higher Mathematics for Science, Technology and Engineering

Authored By

S. G. Ahmed
Department of Engineering Physics and Mathematics
Faculty of Engineering
Zagazig University
Egypt

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PREFACE

First of all let us believe and recall the neglected fact in our life that "***the mother of all human science is Mathematics***". Dear students, readers, researchers and all those who respect mathematics, we introduce this book which may be helpful for them. Most of the researchers directed their own research work to the numerical analysis, due to the rapid communications and the advances in the computer programming. In the present book, we introduce some mathematical concepts that are widely important and can be considered on the basis of the numerical analysis. We presented the topics in a simple way of presentation and no proofs for theorems, because we decided to introduce the scientific material without complications related to those working on pure mathematics. Our main research work is the computational engineering and applied mathematics and numerical analysis; therefore, the present book can serve as the base of a future textbook in numerical analysis. We hope and pray that this work will last forever.

CONFLICT OF INTEREST

The authors confirm that they have no conflict of interest to declare for this publication.

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S. G. Ahmed
Department of Engineering Physics and Mathematics
Faculty of Engineering
Zagazig University
Egypt

Biblography

S. G. Ahmed was born in 1962 in Sharkia, Egypt. Since 2005, he is the Professor of Applied and Engineering Computational Mathematics and Numerical Analysis, in the Department of Engineering Mathematics and Mathematics, at the Zagazig University, Egypt. He was employed at the same department and college since 1985. He received his Ph.D. in Applied and Engineering Computational Mathematics and Numerical Analysis from the University of Portsmouth and Zagazig University a through channel system between UK and Egypt in 1996. He worked as a director of Computational Mechanical Center at Zagazig University, between 2007 and 2010. His present research interest is computational Engineering Mathematics and Numerical Analysis related to different branches of Engineering. He published 51 scientific papers and 8-books in English and Arabic languages. He contributed to more than 50 technical reports. He presented introductory invited talks at more than 10 international conferences. He received the following awards and recognitions: 2006-2016 Who is Who in the world, from USA; EMERALD Literati Highly commended author award, 2006; Symbol of Excellency from Cambridge University, U.K., 2011; and ABI: Scientific Award of Excellency, USA, 2011. He is currently a member of the editorial boards of different international journals with impact factor and active member for different international conferences.

CHAPTER 1

Functions of One Variables

Abstract: This chapter aims mainly to spot the lights on the basis of mathematics of function of one variable, different types, properties. Functions encountered nearly in everything in our daily live and even who works in the field of numerical analysis needs to be in a solid background of the mathematics of functions.

Keywords: Domain, elementary basic functions, function definition, range.

1. INTRODUCTION

Functions arise in a great variety of situations, here are some examples, such as the area of the circle of a given radius, the surface area of a sphere and the volume of the sphere. The functions are also found in different branches of science and also can be found in our usual daily life. One can say, studying the functions are not restricted to mathematics but also in so many branches of science, technology and engineering [1].

2. CALCULUS IN GENERAL

The following diagram shown in Fig. (**1**), is general layout of calculus during this stage of study.

3. FUNCTION DEFINITION

A function is defined as a relation between one or more independent variables and other dependent variable [2]. One important manner for the

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reader is to differentiate between three different topics, function, relation and equation. I see the relation is the more general topic from which one can deduce the other two topics.

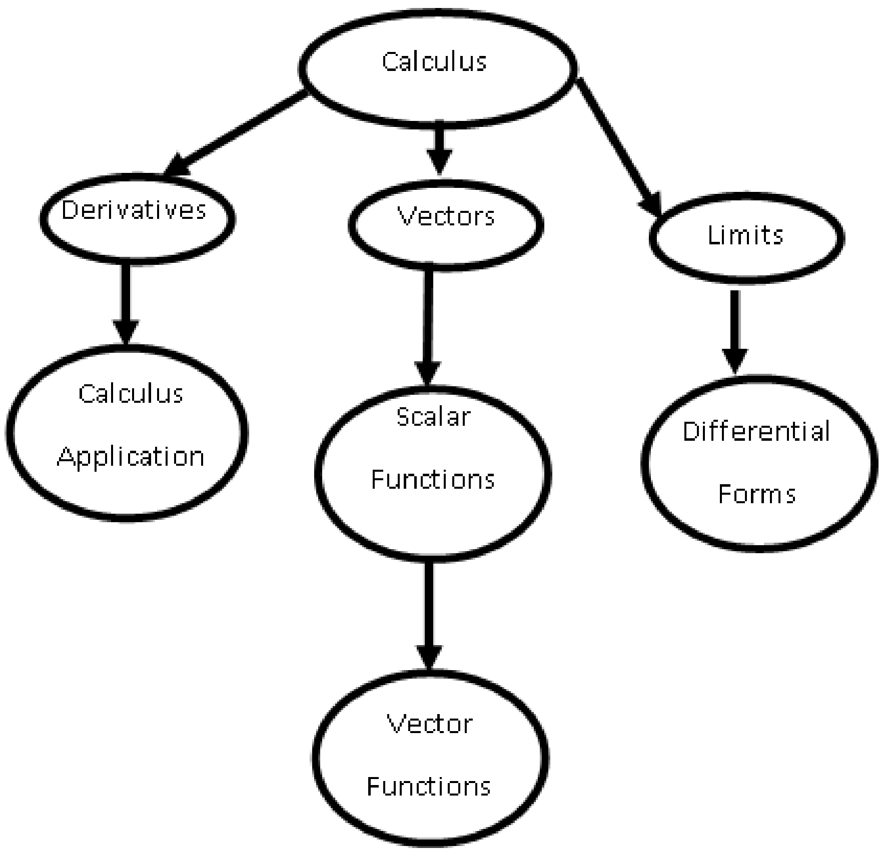

Fig. (1). Calculus branches.

4. CONSTANTS AND VARIABLES

4.1. Constant

A constant is a quantity, which has one and only one value.

Example

The diameter and length of the circumference of a circle can attain different values depending on circumference. Consequently, generally speaking, variables, whereas the ratio of the length of the circumference to its diameter is a constant and equal to π [3].

4.2. Variable

A dependent variable y is a function of a variable x if they are related so that to each value attained by x there corresponds a unique value of the other variable y [4].

Example

For the following, determine if y can be considered a function or not.

(1) $y = 3 - x^2$

(2) $y^2 = 3 - x$

Solution

(1) Referring to the given formula, one can see easily that the exponent of the dependent variable y is one, while the exponent of the independent variable x is two, so for each two symmetric points over the horizontal axis, only single value over the vertical axis. So this equation is said to be a function.

(2) Let us re-write the equation as $y = \pm\sqrt{3-x}$. Here, it is clear that for each value of x there exist two values for y which, is different from the basic definition of a single-valued function. So the given equation is a relation.

CHAPTER 2

Derivatives and Their Applications

Abstract: The basis of the derivative for functions of one variable is introduced herein. The chapter started from the basic definition of the derivative, geometric meaning and growing up to the derivatives of the basic functions. The rules of the derivatives are presented in some details so as the reader can be easily familiar with the different rules of derivatives.

Keywords: Derivatives, domain, elementary basic functions, range.

1. DEFINITION OF DERIVATIVE

The first derivative of any function $f(x)$ is defined as [19]:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{1}$$

The fraction in the right hand side is sometimes called *Fermat's difference quotient* [20].

Example

Find the first derivative for $f(x) = x^2$

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

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$$f^{/}(x) = \lim_{\Delta x \to 0} \frac{\left((x+\Delta x)^2 - x^2\right)}{\Delta x}$$

$$f^{/}(x) = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x}$$

$$= 2x$$

Example

Find the first derivative for $f(x) = x^3 - 3x$ at $x = 2$.

Solution

By definition

$$f^{/}(x) = \lim_{\Delta x \to 0} \frac{\left((x+\Delta x)^3 - 3(x+\Delta x)\right) - \left(x^3 - 3x\right)}{\Delta x}$$

$$f^{/}(x) = 3x^2 - 3$$

Then

$$f^{/}(x) = 3(2)^2 - 3 = 9$$

2. RIGHT AND LEFT HAND DERIVATIVES

A function $f(x)$ is said to be differentiable at a point $x = a$ if the right and the left-hand derivatives at this point are existed [21].

Left-hand derivative

$$f^{/}(a^{-}) = \lim_{h\to 0^{-}} \frac{f(a+h)-f(a)}{h} \qquad \textbf{(2)}$$

Right-hand derivative

$$f^{/}(a^{+}) = \lim_{h\to 0^{+}} \frac{f(a+h)-f(a)}{h} \qquad \textbf{(3)}$$

Example

Discuss the differentiability of the following function:

$$f(x)=\begin{cases} x & x \le 2 \\ x^2 - 2 & x \ge 2 \end{cases} \quad \text{at} \quad x = 2$$

Solution

Let us now evaluate both left and right hand derivatives respectively as follows:

Left-hand derivative

$$f^{/}(a^{-}) = \lim_{h\to 0^{-}} \frac{f(a+h)-f(a)}{h}$$

$$f^{/}(2^{-}) = \lim_{h\to 0^{-}} \frac{x-2}{x-2} = 1$$

Right-hand derivative

CHAPTER 3

Partial Differentiation and Their Applications

Abstract: The basis of the partial derivatives is the basis of the partial differential equations and their wide applications. The chapter started from the basic definition of the partial derivative, geometric meaning and their rules.

Keywords: Domain, range, elementary basic functions, functions of several variables, partial derivatives.

1. INTRODUCTION

In mathematics, a partial derivative of a function of several variables is its derivative with respect to one of those variables with the others held constant. The function $f(x, y, ...)$ has a number of partial derivatives equals the number of its independent variables. The mathematical symbol of the partial derivative is $\frac{\partial}{\partial(.)}$, for example the partial derivative of the function with respect to the independent variable x written as $\frac{\partial f}{\partial x}$ [31].

The basic formula for the first partial derivative is the same as in the total derivative, therefore, let us now define the basic formula for the first partial derivative for the function with respect to the independent variable x, as follows:

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$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \quad \textbf{(1)}$$

The same definition is for any other independent variable:

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \quad \textbf{(2)}$$

2. RULE FOR FINDING PARTIAL DERIVATIVES

To find f_x deal with other independent variables as constant and the same basic rule is still valid for all other independent variable.

Example

Using the basic definition of partial differentiation to find f_x and f_y for $f(x,y) = xy^2$

Solution

By definition;

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)y^2 - xy^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x y^2}{\Delta x} = y^2$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{x(y+\Delta y)^2 - xy^2}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2xy\Delta y}{\Delta y} = 2xy$$

The partial derivatives f_x and f_y are denoted by:

$$f_x = \frac{\partial}{\partial x} f(x, y)$$

$$f_y = \frac{\partial}{\partial y} f(x, y)$$

Example

Find the following partial derivatives.

Find $f_x(2,1)$ and $f_y(2,1)$ given $f(x, y) = x^3 + x^2 y^3 - 2y^2$

Solution

Holding y constant and differentiating with respect to x, we get

$f_x(x, y) = 3x^2 + 2xy^3$ and

So

$$f_x(x = 2, y = 1) = 3x^2 + 2xy^3$$
$$= 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2 y^2 - 4y$$

So

$$f_y(x = 2, y = 1) = 8$$

CHAPTER 4

Fundamentals of Equations and Equation Theory

Abstract: Equations arise in different branches of mathematics, and so the need of dealing with theory of equations as a separate topic of mathematics. In this chapter, the theory of equations are described and explained well and in a brief details. Operations related to equations and transformations are also explained well with detailed examples.

Keywords: Cubic equation, roots of polynomials, remainder theory, synthetic division, theory of equations, transformations of equations.

1. INTRODUCTION

Let us start the subject of ***theory of equations*** by asking our-self the following question: What does it mean by a ***polynomial*** of degree "n" [38]?

$$P(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \quad \textbf{(1)}$$

is called a polynomial of degree n.

Example

$$2 + 3x - 3x^2 + 4x^4 = 0$$

And

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$$x^3 - \sqrt{2}x + \frac{1}{3} = 0$$

The coefficients $a_o,\ a_1,\ a_2, \cdots,\ a_n$ may be complex, such as: $x^4 - i\sqrt{3}x + 8 = 0$

Different types of polynomials given below, as shown in Table **1**;

Table (1). Types of polynomials.

n	**Type**	**Example**
0	*Const.nt*	$P(x) = a_o$
1	*Linear*	$P(x) = a_o + a_1 x$
2	*Quad.atic*	$P(x) = a_o + a_1 x + a_2 x^2$
3	*Cubic*	$P(x) = a_o + a_1 x + a_2 x^2 + a_3 x^3$

2. A POLYNOMIAL ROOTS

An algebraic equation of degree n of the form (1) has exactly n roots. These roots may be *real*, *complex distinct* or *repeated* [39].

3. THEOREM OF THE REMAINDER

The Remainder Theorem is useful for evaluating polynomials at a given value of x. The Theorem talks about dividing the polynomial by some linear factor $(x - a)$, where "a" is just some number. Then, thinking about the long division, you end up with some polynomial answer q(x) (for "quotient polynomial") and some polynomial remainder r(x) [40].

Example

Divide the polynomial p(x) = $x^3 - 7x - 6$, by the linear factor x – 4:

$$
\begin{array}{r|l}
 & x^2 + 4x + 9 \\
\hline
x-4 & x^3 + 0x^2 - 7x - 6 \\
 & -\underline{x^3 + 4x^2} \\
 & \quad 4x^2 - 7x - 6 \\
 & \quad -\underline{4x^2 + 16x} \\
 & \qquad 9x - 6 \\
 & \qquad -\underline{9x + 36} \\
 & \qquad\qquad 30
\end{array}
$$

The final result is $x^2 + 4x + 9$ on top (this is q(x)), and a remainder of 30.

4. THE SYNTHETIC DIVISION

Synthetic division is another way to divide a polynomial by the binomial *x-c*, where *c* is a constant [41]. The major steps can be stated as follow.

Step 1: Set up the synthetic division
Make sure first that you write it in descending powers and you put zero for the missing terms. For example, if you had the problem

$$\left(x^4 - 3x + 5\right) \div (x - 4)$$

The polynomial $x^4 - 3x + 5$, starts out with the fourth degree. This polynomial is missing degrees three and two. We can now write the division as follows:

$$x - 4\overline{\big)\,x^4 + 0x^3 + 0x^2 - 3x + 5}$$

Then

CHAPTER 5

Theory of Determinants and Matrices and their Applications in Linear Equation Theory

Abstract: Determinants and matrices are very important subjects that are widely found in different branches and topics in mathematics. The present chapter deals with these two topics in some brief details starting the preliminary definitions of determinants and matrices, their properties. The linear system of equations from its mathematical constitution till different methods of solutions is also discussed in brief details.

Keywords: Cofactors, determinants, elementary matrices, eigenvectors, Laplace expansion, minors, matrix operations.

1. THE DETERMINANTS

Consider the following system of two linear equations:

$$a_1 x + b_1 y = 0 \quad \textbf{(1-a)}$$

$$a_2 x + b_2 y = 0 \quad \textbf{(1-b)}$$

Eliminating of x and y gives:

$$\frac{a_1}{b_1} = -\frac{y}{x} = \frac{a_2}{b_2} \quad \textbf{(2)}$$

$$a_1 b_2 - a_2 b_1 = 0 \quad \textbf{(3)}$$

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or

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \textbf{(4)}$$

Equation (4) is called a *determinant of second order* [47]. Generally the determinant of order n is written as:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & ..... & l_1 \\ a_2 & b_2 & c_2 & d_2 & ..... & l_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & b_n & c_n & d_n & ..... & l_n \end{vmatrix} \quad \textbf{(5)}$$

The diagonal which contains the elements, $a_1, b_2, c_3, ..., l_n$ is called the major, *leading or principal diagonal.*

2. SOME FUNDAMENTALS

2.1. The Minor

Each element ib any determinant has a minor. This minor is obtained by deleting the row and the column corresponding to that element [48].

Consider the following determinant:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \textbf{(6)}$$

The minor of b_3 is:

$$B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad \textbf{(7)}$$

2.2. The Cofactor

The cofactor is a minor determinant corresponding to any element of the whole determinant, taking the sign rule into consideration. The sign of an element is the *i*th row and *j*th column is $(-1)^{i+j}$. The cofactor of an element is usually denoted by the corresponding capital letter.

2.3. The Laplace Expansion

Laplace expansion is a method to evaluate any determinant. The determinant can be expanded in terms of any row (or column) as follows:

"Multiply each element of the row (or column) by its cofactor and then add up all these terms" [49].

Expand the determinant given in (6), by the first row:

$$\Delta = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \qquad \textbf{(8)}$$

Thus, Δ is the sum of all products of the elements of any row (or column) by the corresponding cofactors.

Example

Evaluate the following determinant:

$$A = \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \\ 8 & 0 & 1 \end{vmatrix}$$

Solution

Expand through the first row:

CHAPTER 6

Partial Fractions

Abstract: Partial fractions encountered within theoretical and applied problems. Also, the partial fractions are good mathematical tools when solving complicated integrals. The main usage of the partial fractions can be considered as a simplified mathematical tool.

Keywords: Fraction decomposition, rational fractions.

1. INTRODUCTION

In algebra, the **partial fraction** decomposition or **partial fraction expansion** is used to reduce the degree of *either* the numerator or the denominator of a rational function. Partial fractions are used in calculating the inverse of transforms; such as the Laplace transform, or the Z-transform [55].

2. DEFINITION OF THE PROPER FRACTION

A FUNCTION OF THE FOLLOWING FORM [56]:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0 \qquad \textbf{(1)}$$

Where $a_n, a_{n-1}, a_{n-2}........, a_1$, and a_0 are constants and n is a positive integer, is said to be a *polynomial* in *x* in degree *n., e.g.* $p(x) = 5x^2 + +2x - 7$ is a polynomial of second degree.

3. RATIONAL FRACTION

A rational function is a quotient polynomial function, that is [57],

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$$f(x) = \frac{P(x)}{Q(x)} \qquad \textbf{(2)}$$

Where $P(x)$ and $Q(x)$ are two polynomials in *x*.

For example:

$$f(x) = \frac{x^4 - 3}{x^2 + 2x + 1}$$

is a rational function. A rational function is ***improper*** if the degree of the numerator is greater than the denominator and other wise is ***proper***.

Example

Consider the following two fractions:

(1) $$\frac{3x^2 - 1}{x^3 + 7x^2 - 4}$$

(2) $$\frac{2x^3 + 6x^2 - 9}{x^2 - 3x + 2}$$

The first one is a proper fraction because the maximum degree of the numerator is less than the maximum degree of the denominator. The second one is an improper fraction, as the degree of numerator is larger than the degree of the denominator. To reduce an improper fraction to a proper one, the numerator is divided by the denominator.

The result of this process is

$$(2x + 13)$$

And a reminder

$$\left(\frac{32x-33}{x^2-3x+2}\right)$$

Which is in fact a *proper* fraction as the degree of the numerator is smaller than the degree of denominator.

$$\frac{2x^3+6x^2-9}{x^2-3x+2}=2\text{x}+12+\frac{32\text{x}-33}{\text{x}^2-3x+2}$$

FINDING ROOTS OF THE DENOMINATOR IN A FRACTIONAL FUNCTION

The most important step in partial fraction is to find the roots of the denominator. Here are some examples.

Example

$$\frac{x+4}{(x^2-4)(x+1)}=\frac{x+4}{(x-2)(x+2)(x+1)}$$

Example

Expand the denominator in

$$\frac{x+5}{x^6-1}=\frac{x+5}{(x^3-1)(x^3+1)}$$

$$=\frac{x+5}{(x-1)(x^2+x+1)(x+1)(x^2-x+1)}$$

Example

$$\frac{x+8}{x(x^4-16)} = \frac{x+8}{x(x^2-4)(x^2+4)}$$
$$= \frac{x+8}{x(x-2)(x+2)(x^2+4)}$$

Example

$$\frac{\mathrm{x}+4}{x^2+13\mathrm{x}-7} = \frac{\mathrm{x}+4}{(\mathrm{x}+7)(2\mathrm{x}-1)}$$

PARTIAL FRACTIONS

Case of Linear Denominators

In the case the denominator p(x) is a multinomial function (*i.e.* a function of x^n with many elements) and it is possible to split this function into several linear factors. In this case the rational function writes:

$$\frac{f(x)}{p(x)} = \frac{f(x)}{(x-x_1)(x-x_2).....(x-x_n)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + ...\frac{A_n}{x-x_n} \qquad \textbf{(3)}$$

The coefficients $A_1, A_2, \ldots A_n$ are evaluated as follows

<u>First Method</u>

1-Reduce equation (3) into the same denominator:

$$\begin{aligned}&A_1(x-x_2)(x-x_3).......(x-x_n)\\&+A_2(x-x_1)(x-x_3).......(x-x_n)\\&.....\\&A_n(x-x_1)(x-x_2).......(x-x_{n-1})\end{aligned} \quad \textbf{(4)}$$

2-Equate $x=x_1$ in eq.(4), gives $A_1 = f(x)/(x-x_2)(x-x_3)..(x-x_n)$

Equate $x=x_2$ in eq.(4), gives $A_2 = f(x)/(x-x_1)(x-x_3)..(x-x_n)$

Similarly for the coefficients $A_3, A_4, \ldots A_n$

<u>Second Method</u>

The coefficients are evaluated directly as :

$$\begin{aligned}A_1 &= \lim_{x \to x_1} \frac{f(x)}{(x-x_2)(x-x_3)...(x-x_n)}\\A_2 &= \lim_{x \to x_2} \frac{f(x)}{(x-x_1)(x-x_3)...(x-x_n)}\\&.....\\A_n &= \lim_{x \to x_n} \frac{f(x)}{(x-x_1)(x-x_2)...(x-x_{n-1})}\end{aligned} \quad \textbf{(5)}$$

Example

Expand the function $\dfrac{x+4}{x^2+13x-7}$

Solution

The function $\dfrac{x+4}{x^2+13x-7}$ is a proper fraction and is expanded as

$$\frac{x+4}{x^2+13x-7}$$

$$=\frac{x+4}{(x+7)(2x-1)}=\frac{A}{x+7}+\frac{B}{2x-1} \qquad \text{(a-1)}$$

Evaluation of the coefficients A and B

First method

Reduce equation (a-1) to the same denominator

$$A(2x-1)+B(x+7)=x+4$$

at x=1/2 B(1/2+7) =1/2+4

Get

B=9/15

At x=-7 A[2(-7)-1]=-7+4

A(-15) = -3,

We get

A= 1/5

Hence

$$\frac{x+4}{x^2+13x-7}=\frac{1}{5}\frac{1}{x+7}+\frac{9}{15}\frac{1}{2x-1}$$

Second method

$$\frac{x+4}{x^2+13x-7} = \frac{x+4}{(x+7)(2x-1)}$$
$$= \frac{A}{x+7} + \frac{B}{2x-1}$$

$$A = \lim_{x->-7} \frac{x+4}{2x-1}$$
$$= \frac{-7+4}{-15}$$
$$= \frac{1}{5}$$

$$B = \lim_{x->\frac{1}{2}} \frac{x+4}{x+7}$$
$$= \frac{\frac{1}{2}+4}{\frac{1}{2}+7} = \frac{9}{15}$$

Both results are the same.

Example

Find the partial fraction expansion of

$$\frac{7x^2 - 25x + 6}{(x^2 - 2x - 1)(3x - 2)} \qquad \text{(a-1)}$$

Solution

Roots of

$(x^2-2x-1)(3x-2)=0$

$$1-\textit{First parenthis}$$
$$x^2-2x-1=0 \qquad \text{(a-2)}$$
$$x_{1,2}=1\pm\sqrt{1+1}$$
$$=1\pm\sqrt{2}$$
$$2-\textit{Second parenthis}$$
$$3x-2=o$$
$$x_3=2/3$$

Expand the fraction into partial fractions

$$\frac{7x^2-25x+6}{(x^2-2x-1)(3x-2)}=\frac{Ax+B}{(x^2-2x-1)}+\frac{C}{3x-2} \qquad \text{(a-3)}$$

Notice that the quadratic denominator is kept as $x_{1,2}$ are irrational roots.

The coefficient of the second term is:

$$C=\lim_{x\to\frac{2}{3}}\frac{7x^2-25x+6}{(x^2-2x-1)}$$
$$=4$$

Then reduce equation (a-1) to the same denominator

$(Ax+B)(3x-2)+C(x^2-2x-1)=7x^2-25x+6$

Equate coefficients of different powers of *x*

Coefficient x^2 $3A + C = 7$

Replace for C = 4 $\therefore\ 3A = 3$

A = 1

Coefficient of x^1 $3B – 2A – 2C = -25$

$3B = -15$

we get

B = -5

Replace for A,B and C in eq.(a-1)

$$\frac{7x^2 - 25x + 6}{(x^2 - -1)(3x - 2)} = \frac{x - 5}{x^2 - 2x - 1} + \frac{4}{3x - 2}$$

RATIONAL FUNCTIONS WITH REPEATED LINEAR DENOMINATOR

$$\frac{f(x)}{(ax+b)^n} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots.. + \frac{A_n}{(ax+b)^n} \qquad \textbf{(6)}$$

The coefficients $A_1, A_2, \ldots. A_n$ are evaluated by one of the two methods:

First method

Eq.(6) is reduced to the same denominator $(ax+b)^n$

$$A_1(ax+b)^{n-1} + A_1(ax+b)^{n-1} + \ldots. + A_n = f(x) \qquad \textbf{(7)}$$

The coefficients of x^0, x^1 , x^2 , ….x^{n-1} are equated in both sides od eq.(7) .

This results in a system of (n) equations in the

unknown $A_1, A_2, \ldots A_n$, which are solved subsequently.

Second method

Is known as Heaviside method, where the coefficients $A_1, A_2, \ldots A_n$ are evaluated in a descending order according to the following formulas

$$A_n = \lim_{x->-b/a} f(x)$$

$$A_{n-1} = \lim_{x->-b/a} \frac{d}{dx} f(x)$$

$$A_n = \lim_{x->-b/a} \frac{1}{2!} \frac{d^2}{dx^2} f(x) \qquad \textbf{(8)}$$

..

$$A_n = \lim_{x->-b/a} \frac{1}{(n-1)!} \frac{d^n}{dx^n} f(x)$$

Example

Express $\frac{x^2 - 4x - 15}{(x+2)^3}$ in a partial fraction form

Solution

$$\frac{x^2 - 4x - 15}{(x+2)^3} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \qquad \text{(a-1)}$$

where A, B, C are calculated using Heaviside method

$$A = \lim_{x \to -2} \frac{1}{2!} \frac{d}{dx^2} (x^2 - 4x - 15) \qquad \text{(a-2)}$$

$$A = \lim_{x\to -2} \frac{2}{2!} = 1 \qquad \text{(a-3)}$$

$$B = \lim_{x\to -2} \frac{d}{dx} f(x) = \lim_{x\to -2} 2x - 4 = -8 \qquad \text{(a-4)}$$

$$C = \lim_{x\to -2} f(x) = (-2)^2 - 4(-2) - 15 = 3 \qquad \text{(a-5)}$$

Replace for A, B, C in the original function:

$$\therefore \frac{x^2 - 4x - 15}{(x+2)^3} = \frac{1}{x+2} - \frac{8}{(x+2)^2} - \frac{3}{(x+2)^3} \qquad \text{(a-6)}$$

Another method of solution

$$\frac{x^2 - 4x - 15}{(x+2)^3} \text{ let } y = x + 2, \quad \text{then } x = y - 2 \qquad \text{(b-1)}$$

$$\therefore \frac{x^2 - 4x - 15}{(x+2)^3} = \frac{(y-2)^2 - 4(y-2) - 15}{y^3}$$

$$= \frac{y^2 - 8y - 3}{y^3}$$

$$= \frac{1}{y} - \frac{8}{y^2} - \frac{3}{y^3} \qquad \text{(b-2)}$$

$$= \frac{1}{x+2} - \frac{8}{(x+2)^2} - \frac{3}{(x+2)^3}$$

which is the result obtained previously.

Example

$$\frac{x^2-6x+2}{x^2(x-2)^2}=\frac{A}{x^2}+\frac{B}{x}+\frac{C}{(x-2)^2}+\frac{D}{x-2}$$

Solution

Heaviside method is used

Roots of the denominator $x^2(x-2)^2=0$

The roots x = 0, x = 2 are double

$$A=\lim_{x\to 0}\frac{x^2-6x+2}{(x-2)^2}=\frac{2}{-2^2} \qquad \text{(c-1)}$$

$$A=\frac{1}{2} \qquad \text{(c-2)}$$

Similarly

$$B=\lim_{x\to 0}\frac{d}{dx}\frac{f(x)}{(x-2)^2}$$

$$=\lim_{x\to 0}\frac{f'(x)(x-2)^2-2f(x)(x-2)}{(x-2)^4} \qquad \text{(c-3)}$$

$$=\lim_{x\to 0}\frac{f'(x)(x-2)-2f(x)}{(x-2)^3}$$

where $f(x)=x^2-6x+2$, $f'(x)=2x-6$

we get

$$B = \frac{(-6)(-2) - 2^2}{-2^3} = -1 \qquad \text{(c-4)}$$

$$C = \lim_{x \to 2} \frac{f(x)}{x^2} = \frac{f(2)}{4} = \frac{-6}{4} = \frac{-3}{2} \qquad \text{(c-5)}$$

$$\begin{aligned} D &= \lim_{x \to 2} \frac{d}{dx} \frac{f(x)}{x^2} \\ &= \lim_{x \to 2} \frac{f'(x).x^2 - 2xf(x)}{x^4} \\ &= \lim_{x \to 2} \frac{f'(x).x - 2f(x)}{x^3} \qquad \text{(c-6)} \\ &= \frac{f'(2).2 - 2f(2)}{2^3} \\ &= \frac{(-2)(2) - 2(-6)}{8} = 1 \end{aligned}$$

Replace for A, B, C, D in the original equation

$$\frac{x^2 - 6x + 2}{x^2 (x - 2)^2} = \frac{1}{2x^2} - \frac{1}{x} - \frac{3}{2(x-2)^2} + \frac{1}{(x-2)} \qquad \text{(c-7)}$$

Example

Express in partial fraction

$$\frac{2x^2 + 7x + 23}{(x-1)(x+3)^2} = \frac{A}{(x-1)} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)}$$

Solution

$$A = \lim_{x\to 1} \frac{2x^2 + 7x + 23}{(x-3)^2} = \frac{2+7+23}{(4)^2} = \frac{32}{16} = 2 \qquad \text{(d-1)}$$

By Heaviside method

$$B = \lim_{x\to -3} \frac{2x^2 + 7x + 23}{(x-1)}$$
$$= \frac{2.9 - 7.3 + 23}{-4} \qquad \text{(d-2)}$$
$$= \frac{18 - 21 - 23}{-4}$$
$$= -5$$

$$C = \lim_{x\to -3} \frac{d}{dx} \frac{f(x)}{(x-1)}$$
$$= \lim_{x\to -3} \frac{f'(x)(x-1) - f(x)}{(x-1)} \qquad \text{(d-3)}$$

Where

$f(x) = 2x^2 + 7x + 23$ (d-4)

$f(-3) = 2.9 - 7.3 + 23 = 20$ (d-5)

$f(x) = 4x + 7$ (d-6)

$f'(-3) = 12 + 7 = -5$ (d-7)

$$\therefore C = \frac{f'(-3)(-4) - (-3)}{-4} = \frac{(-5)(-4) - 20}{-4} = 0 \qquad \text{(d-8)}$$

Replace for A,B,C, and D in the original function;

$$\therefore \frac{2x^2 + 7x + 23}{(x-1)(x+3)^2} = \frac{2}{x-1} - \frac{5}{(x+3)^2} \qquad \text{(d-9)}$$

CASE OF A QUADRATIC FACTOR IN THE DENOMINATOR

$$\frac{f(x)}{p(x)} = \frac{f(x)}{(ax^2+bx+c)(x-x_1)} = \frac{Ax+B}{(ax^2+bx+c)} + \frac{C}{(x-x_1)} \qquad \textbf{(9)}$$

In this case C is evaluated by :

$$C = \lim_{x \to x_1} \frac{f(x)}{ax^2+bx+c} \qquad \textbf{(10)}$$

And A, B are obtained from the reduction of equation (4) to the same denominator; and the comparison of the coefficients of x^0, x^1,..x^n on both sides of the following equation:

$$(Ax+B)(x-x_1) + C(ax^2+bx+c) = f(x) \qquad \textbf{(11)}$$

Example

Expand in a partial fraction form

$$\frac{4x^2 - 28}{x^4 + x^2 - 6} \qquad \text{(a-1)}$$

Solution

$$\frac{4x^2-28}{x^4+x^2-6}=\frac{4x^2-28}{(x^2+3)(x^2-2)} =\frac{Ax+B}{x^2+3}+\frac{Cx+D}{x^2-2} \qquad \text{(a-2)}$$

Reduce eq. (a-2) to the same denominator

$(Ax-B)(x^2-2)+(Cx+D)(x^2+3)= 4\,x^2-28$

Equate

Coefficient of x^3 $A+C= 0 \therefore A= -C$ (a-3)

Coefficient of x^2 $B+D= 4 \therefore D=4-B$ (a-4)

Coefficient of x $-2A +3C = 0$ (a-5)

Replace for $A=-C$ get $5C = 0 \therefore A=C=0$ (a-6)

Coefficient of x^o $- 2B+3D= -28$ (a-7)

Solving:

$5D= -20 \therefore D= -4$ (a-8)

$B=8$ (a-9)

Replace for A,B,C and D, we get:

$$\frac{4x^2-28}{x^4+x^2-6}=\frac{8}{x^2+3}-\frac{4}{x^2-2} \qquad \text{(a-10)}$$

RATIONAL FUNCTION WITH REPEATED QUADRATIC ROOTS

$$\frac{f(x)}{p(x)}=\frac{f(x)}{\left(ax^2+bx+c\right)^k}$$
$$=\frac{A_1x+B_1}{\left(ax^2+bx+c\right)}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+......+\frac{A_kx+B_k}{\left(ax^2+bx+c\right)^k} \qquad \textbf{(12)}$$

Example (12)

Find the partial fraction for $\frac{2x^2+10x-3}{(x+1)(x^2-9)}$

Solution

Let the denominator

$2x^3 + 10x - 3 = f(x)$ (a-1)

$$\frac{f(x)}{(x+1)(x^2-9)}=\frac{A}{x+1}+\frac{B}{x-3}+\frac{C}{x+3} \qquad \text{(a-2)}$$

Get *A*, *B* and *C* as follows

$$A=\lim_{x\to-1}\frac{f(x)}{(x^2-9)}=\frac{2(1)-10-3}{1-9}=\frac{-11}{-8}=\frac{1}{8} \qquad \text{(a-3)}$$

$$B=\lim_{x\to3}\frac{2x^2+10x-3}{(x+1)(x+3)}=\frac{2.(3)^2+10x3-3}{4.6}=\frac{5}{8} \qquad \text{(a-4)}$$

Similarly C is evaluated

$$C = \lim_{x \to -3} \frac{2x^2 + 10x - 3}{(x - 1)(x - 3)} = \frac{2.(3)^2 + 10(-3) - 3}{(-3 - 1)(-3 - 3)} = \frac{-5}{8} \qquad \text{(a-5)}$$

$$\frac{2x^2 + 10x - 3}{(x + 1)(x^2 - 9)} = \frac{1/8}{x + 1} + \frac{5/8}{x - 3} - \frac{5/8}{x + 3} \qquad \text{(a-6)}$$

Example

Express $\dfrac{1}{x^3 - x^2 - x + 1}$ as partial fractions

Solution

$$\frac{1}{x^2(x - 1) - (x - 1)} = \frac{1}{(x^2 - 1)(x - 1)} = \frac{1}{(x - 1)^2(x + 1)} \qquad \text{(b-1)}$$

This form is the first step to express fraction in a partial fraction form. It is required to complete the solution.

Example

Find partial fraction Decomposition of

$$f(x) = \frac{3x - 5}{(x - 1)\,(x\text{-}2)}$$

Solution

$$\frac{3x - 5}{(x - 1)\,(x\text{-}2)} = \frac{A}{(x - 1)} + \frac{B}{(x - 2)}$$

Reducing to the common denominator and equating the numerator, we have:

$$3x-5=\mathrm{A}(\mathrm{x}-2)+B(x-1)$$

There are two methods to find the value of *A* and *B:*

Method (1)

Put *x*=1 in the above equation, we get:

$$3(1)-5=\mathrm{A}(1-2) \quad , \quad \therefore A=2$$

Put *x*=2 in the above equation also, we get:

$$3(2)-5=B(2-1) \quad , \quad \therefore B=1$$

Method (2)

Equating the coefficients of $x^1 \text{and } x^0$ (absolute term), we get

a system of equations for determining the unknown coefficients:

$3=A+B$, and $5=2A+B$,

then solving this system of equations, we get:

$$A=2, \text{and } B=1$$

$$\therefore \frac{3x-5}{(x-1)\,(x\text{-}2)}=\frac{2}{(x-1)}+\frac{1}{(x-2)}$$

Example

Find partial fraction Decomposition of

$$f(x) = \frac{x+4}{2x^2+13x-7}$$

Solution

$$f(x) = \frac{x+4}{2x^2+13x-7} = \frac{x+4}{(x+7)(2x-1)} = \frac{A}{(x+7)} + \frac{B}{(2x-1)}$$

Reducing to the common denominator and equating the numerator, we have:

$$x+4 = A(2x\text{-}1) + B(x+7)$$

At $x = \frac{1}{2}$ $\rightarrow B\left(\frac{1}{2}+7\right) = \left(\frac{1}{2}+4\right)$, $\therefore B = \frac{9}{15}$

At $x = -7$ $\rightarrow A(2(-7)-1) = -7+4$, $\therefore A = \frac{1}{5}$

$$\therefore \frac{x+4}{2x^2+13x-7} = \frac{\frac{1}{5}}{(x+7)} + \frac{\frac{9}{15}}{(2x-1)}$$

Example

Find partial fraction Decomposition

$$f(x) = \frac{x^2 + 2}{(x+1)^3\ (x\text{-}2)}$$

Solution

$$\frac{x^2 + 2}{(x+1)^3\ (x\text{-}2)} = \frac{A}{(x+1)^3} + \frac{A_1}{(x+1)^2}$$
$$+ \frac{A_2}{(x+1)} + \frac{B}{(x-2)}$$

Reducing to the common denominator and equating the numerator we have:

$$(x^2 + 2) = \mathrm{A(x - 2)} + A_1(x+1)\mathrm{(x - 2)}$$
$$+ A_2(x+1)^2(x-2) + B(x+1)^3$$

Equating the coefficients of $x^3, x^2, x,$ and x^0 (absolute term), we get a system of equations for determining the unknown coefficients,

$$A = -1, A_1 = \frac{1}{3}, A_2 = \frac{-2}{9}, \text{and } B = \frac{2}{9}.$$

$$\frac{x^2 + 2}{(x+1)^3\ (x\text{-}2)} = \frac{-1}{(x+1)^3} + \frac{1}{3(x+1)^2}$$
$$- \frac{2}{9(x+1)} + \frac{2}{9(x-2)}$$

Example

Express $\dfrac{x^4 + 2x^3\text{-}2x^2\text{-}x - 6}{(x^3 - 1)}$ as the sum of partial fractions

Solution

First step is to reduce the given fraction to a proper one by division

$$\frac{x^4+2x^3\text{-}2x^2\text{-}x-6}{(x^3-1)}=(x+2)-\frac{2(x^2+2)}{(x^3-1)}$$

Second step is to reduce the decompose (x^3-1) to simpler form, *i.e.*,

$$(x^3-1)=(x-1)(x^2+x+1)$$

Now we have

$$\frac{x^4+2x^3\text{-}2x^2\text{-}x-6}{(x^3-1)}=(x+2)-\frac{2(x^2+2)}{(x-1)(x^2+x+1)}$$

i.e., we have to reduce the term

$$\frac{2(x^2+2)}{(x-1)(x^2+x+1)}$$

to its partial fractions , *i.e.*,

$$\frac{2(x^2+2)}{(x-1)(x^2+x+1)}=\frac{A}{(x-1)}+\frac{Bx+C}{(x^2+x+1)}$$

Or

$$2(x^2+2)=A(x^2+x+1)+\left(Bx+C\right)(x-1)$$

To obtain *A*, put *x*=1 in the above equation, then; $A=2$.

Coeff. of x^0, give $4 = A\text{-}C$, then $C = \text{-}2$

Coeff. of x^1, give $0 = A\text{-}B + C$, then $B = 0$

$$\frac{2(x^2+2)}{(x-1)(x^2+x+1)} = \frac{2}{(x-1)} - \frac{2}{(x^2+x+1)}$$

$$\frac{x^4+2x^3\text{-}2x^2\text{-}x-6}{(x^3-1)} = (x+2) - 2\left(\frac{1}{(x-1)} - \frac{1}{(x^2+x+1)}\right)$$

Example

Express

$$\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}$$

as the sum of partial fractions

Solution

Decompose the fraction into partial fractions

$$\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}$$

$$= \frac{Ax+B}{(x^2+2x+3)^2} + \frac{Cx+D}{(x^2+2x+3)} + \frac{E}{x+1}$$

Where

$$x^4+4x^3+11x^2+12x+8 = (Ax+B)(x+1)$$

$+(Cx+D)(x^2+2x+3)(x+1)+E(x^2+2x+3)^2$ Combining the above

indicated methods of determining coefficients, we find:
To obtain E, put $x=-1$ in the above equation, then:

$E=1$.

Coeff. of x^4, give $1=C+E$

Coeff. of x^3, give $4=3C+D+4E$

Coeff. of x^2, give $11=A+5C+3D+10E$

Coeff. of x^1, give

$12=A+B+3C+5D+12E$

Coeff. of x^0, give $8=B+3D+9E$

Then solving this system of equations, we get:

$A=1$, $B=-1$, $C=0$, $D=0$, and $E=1$.

Therefore;

$$\frac{x^4+4x^3+11x^2+12x+8}{(x^2+2x+3)^2(x+1)}=\frac{x-1}{(x^2+2x+3)^2}+\frac{1}{x+1}$$

Example

Find the partial fraction expansion of

$$\frac{7x^2-25x+6}{(x^2-2x-1)(3x-2)}$$

Solution

$$\frac{7x^2\text{-}25x+6}{(x^2-2x-1)(3x-2)}=\frac{Ax+B}{(x^2-2x-1)}+\frac{C}{(3x-2)}$$

Where

$7x^2\text{-}25x+6=(Ax+B)(3x-2)+C(x^2-2x-1)$ Equate coefficients of different powers of x, we find:

Coef. of x^2, give $3A+C=7$

Coef. of x^1, give $-2A+3B-2C=-25$

Coef. of x^0, give $-2B-C=6$

Then solving this system of equations, we get : $A=1$, $B=-5$, and $C=4$

$$\therefore \frac{7x^2\text{-}25x+6}{(x^2-2x-1)(3x-2)}=\frac{x-5}{(x^2-2x-1)}+\frac{4}{(3x-2)}$$

SUPPLEMENTARY PROBLEMS

FIND PARTIAL FRACTION DECOMPOSITION OF THE FOLLOWING RATIONAL FRACTIONS:

(1) $\dfrac{2x-1}{(x-1)(x-2)}$

(2) $$\frac{x}{(x+1)(x+3)(x+5)}$$

(3) $$\frac{x+2}{(x^2-7x+12)}$$

(4) $$\frac{2x+2}{(x^2-x-12)}$$

(5) $$\frac{x^2}{(x^2-4)}$$

(6) $$\frac{5x+4}{x^2+2x}$$

(7) $$\frac{x^4}{(x^2-1)(x+2)}$$

(8) $$\frac{3x+2}{x(x+1)^3}$$

(9) $$\frac{1}{x(x^2+1)}$$

(10) $$\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}$$

(11) $$\frac{1}{x^3+1}$$

(12) $$\frac{3x-7}{x^3+x^2+4x+4}$$

(13) $\dfrac{x^3+x-1}{(x^2+2)^2}$

(14) $\dfrac{1}{(x^2-x)(x^2-x+1)}$

(15) $\dfrac{2x^4-x^3+6x^2-x+8}{x(x^2+2)^2}$

(16) $\dfrac{45x^3-42x^2+20x-1}{15x^2-14x+3}$

(17) $\dfrac{x^3-10x^2+13}{(x-1)\left(x^2-5x+6\right)}$

(18) $\dfrac{1}{x^3-x^2-x+1}$

(19) $\dfrac{x+2}{x^2-7x+12}$

(20) $\dfrac{12x+11}{x^2+x-6}$

(21) $\dfrac{8-x}{2x^2+3x-2}$

(22) $\dfrac{x^3}{x^2-4}$

(23) $\dfrac{3x^2-8x+9}{(x-2)^3}$

(24) $\dfrac{3x^3+10x^2+27x+27}{x^2(x+3)^2}$

(25) $\dfrac{5x^2+8x+21}{(x^2+x+6)(x+1)}$

(26) $\dfrac{7x-2}{x^3-x^2-2x}$

(27) $\dfrac{x^3}{(x^2+4)^2}$

(28) $\dfrac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

(29) $\dfrac{2x^3-x+3}{(x^2+4)(x^2+1)}$

(30) $\dfrac{7x^3+16x^2+20x+5}{(x^2+2x+2)^2}$

(31) $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$

(32) $\dfrac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$

(33) $\dfrac{46+13x}{12x^2-11x-15}$

(34) $\dfrac{9}{(x-1)(x+2)^2}$

(35) $$\frac{7x-1}{1-5x^2+16x^2}$$

(36) $$\frac{x^4-3x^3+3x^2+10}{(x+1)^2(x-3)}$$

(37) $$\frac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$$

(38) $$\frac{3x^3-8x^2+10}{(x-1)^4}$$

(39) $$\frac{x^3-10x+13}{(x-1)(x^2-5x+6)}$$

(40) $$\frac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$$

(41) $$\frac{2x-1}{(x-1)(x-2)}$$

(42) $$\frac{x}{(x+1)(x+3)(x+5)}$$

(43) $$\frac{x+2}{(x^2-7x+12)}$$

(44) $$\frac{2x+2}{(x^2-x-12)}$$

(45) $$\frac{x^2}{(x^2-4)}$$

(46) $\dfrac{5x+4}{x^2+2x}$

(47) $\dfrac{x^4}{(x^2-1)(x+2)}$

(48) $\dfrac{3x+2}{x(x+1)^3}$

(49) $\dfrac{1}{x(x^2+1)}$

(50) $\dfrac{2x^2-3x-3}{(x-1)(x^2-2x+5)}$

(51) $\dfrac{1}{x^3+1}$

(52) $\dfrac{3x-7}{x^3+x^2+4x+4}$

(53) $\dfrac{x^3+x-1}{(x^2+2)^2}$

(54) $\dfrac{1}{(x^2-x)(x^2-x+1)}$

(55) $\dfrac{2x^4-x^3+6x^2-x+8}{x(x^2+2)^2}$

(56) $\dfrac{45x^3-42x^2+20x-1}{15x^2-14x+3}$

(57) $\dfrac{x^3 - 10x^2 + 13}{(x-1)\left(x^2 - 5x + 6\right)}$

(58) $\dfrac{1}{x^3 - x^2 - x + 1}$

(58) $\dfrac{x + 2}{x^2 - 7x + 12}$

(59) $\dfrac{12x + 11}{x^2 + x - 6}$

(60) $\dfrac{8 - x}{2x^2 + 3x - 2}$

(61) $\dfrac{x^3}{x^2 - 4}$

(62) $\dfrac{3x^2 - 8x + 9}{(x-2)^3}$

(63) $\dfrac{3x^3 + 10x^2 + 27x + 27}{x^2(x+3)^2}$

CHAPTER 7

Vector Differential and Integral Calculus

Abstract: The topics vector differential and integral calculus arise in many practical and engineering applications. This chapter concerns mainly with these topics but few revision on the basis of vectors is introduced. Scalar and vector functions are also introduced due to their importance in numerical analysis.

Keywords: Inner product of vectors, scalar function, vector definition, vector product, vector function.

1. VECTORS QUANTITIES

A vector describes a physical quantity, such as displacement velocity and acceleration. It is represented by a bold letter **V**, or $\underline{V}$ We will describe in this chapter operations like, addition subtraction, multiplication of vector (dot and cross products). A *direction* and a *magnitude* describes a vector. Fig. (**1**) shows $\vec{A}$& $\vec{B}$ are two vectors with different magnitude and direction.

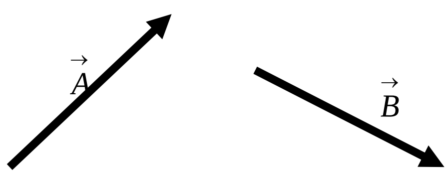

Fig. (1). $\vec{A}$& $\vec{B}$ are two vectors with different magnitude and direction.

1.1. Vector Definition

A vector is a quantity that is determined by both its magnitude and direction. The vector may describes a physical quantity, such as displacement velocity and acceleration [58].

S. G. Ahmed

1.2. Vector Length

The length (Magnitude) of the vector a is also called the norm (or Euclidean norm) of the vector a and is denoted BY $|a|$.

1.3. Equality of Vectors

Two vectors $\vec{A} \& \vec{B}$ are said to be equal if they have the same magnitude and direction.

2. COMPONENTS OF UNIT VECTOR

Any vector has a three components associated with the principle axes as follow [59]:

$$i = [1,0,0] \qquad j = [0,1,0] \qquad k = [0,0,1] \tag{1}$$

These unit vectors are shown in Fig. (**2**), with coordinates on the axes.

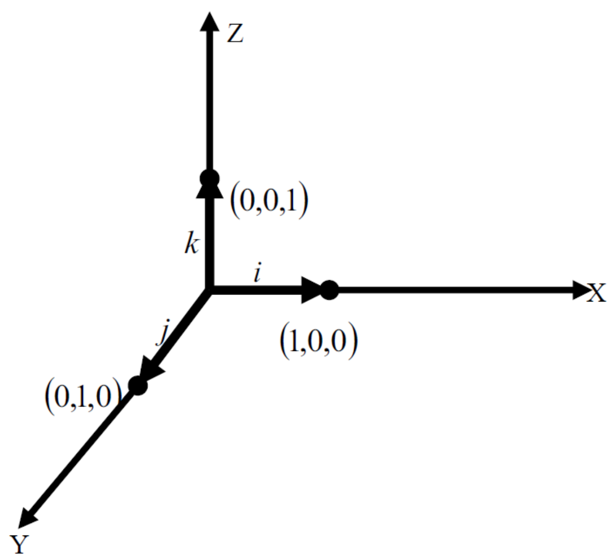

Fig. (2). Unit vector and its components.

3. SPECIAL TYPES OF VECTORS

3.1. Zero – Null – Vector

It is a vector having no direction and zero magnitude and denoted by the symbol O.

3.2. Proper Vector

The proper vector is defined as any vector of any magnitude differs from zero and have defined direction

3.3. Position Vector

A Cartesian coordinate system being given, the position vector r of a point $A:(x, y, z)$ is the vector with the origin $(0,0,0)$ as the initial point and $A:(x, y, z)$ as the terminal point. Thus $r:[x, y, z]$

4. GEOMETRICAL ADDITION AND SUBTRACTION OF VECTORS

To deal with vectors, it is very important to be in a good understanding well the geometric interpretation of the vector, and then we can move after that to addition, subtraction, and multiplication [60]. Let us refer to Fig. (**3**), and assume that we have two vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$

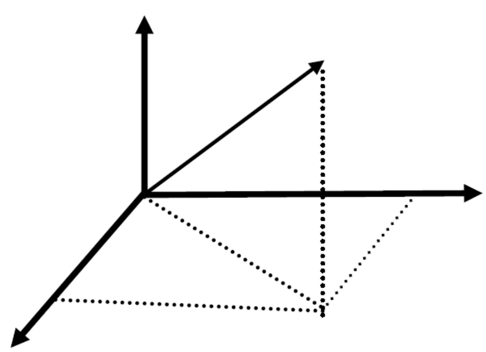

Fig. (3). The position vector and its components.

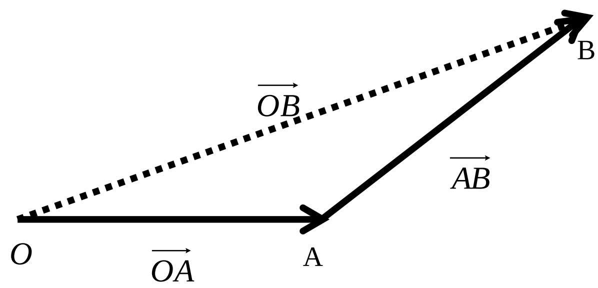

Fig. (4). Vectors addition.

In Fig. (**4**), the vector $\overrightarrow{OB}$ is the resultant vector of the two vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$, respectively. Mathematically, this can be written as:

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} \qquad \textbf{(2)}$$

Equation (3), represents the addition of two vectors. Let us now turn to the subtraction of the same two vectors but in this case, the direction of the two vectors will be opposite to each other, see Fig. (**5**), then the vector $\overrightarrow{OB}$ will represents the subtraction of the two vectors.

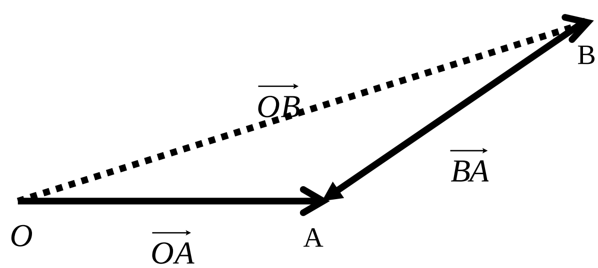

Fig. (5). Vector subtraction.

Mathematically, this subtraction will be written as:

$$\overrightarrow{OB} = \overrightarrow{OA} - \overrightarrow{BA} \qquad \textbf{(3)}$$

Theorem

The commutative and associative laws holds for addition of any number of vectors.

To understand the concept of the commutative and associative laws, let us refer back to Fig. (**6**) and have a look to the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$, and written as a and b, then:

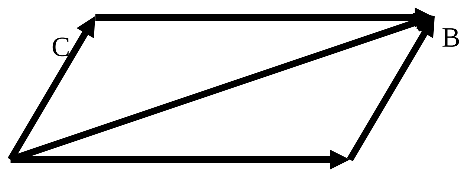

Fig. (6). Addition of two vectors.

$$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = a + b \qquad \textbf{(4)}$$

$$\overrightarrow{OC} = \overrightarrow{AB} = b$$
$$\overrightarrow{CB} = \overrightarrow{OA} = a \qquad \textbf{(5)}$$

Thus

$$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB} = b + a \qquad \textbf{(6)}$$

Therefore

$$a + b = b + a \qquad \textbf{(7)}$$

From equations (6-8), one can prove the theorem.

Refer to Fig. (**7**).

$$\overrightarrow{BD} = c \qquad \textbf{(8)}$$

Then

$$\overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \left(\overrightarrow{OA} + \overrightarrow{AB}\right) + \overrightarrow{BD} = (a+b)+c \quad \textbf{(9)}$$

Also

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{AD} = \overrightarrow{OA} + \left(\overrightarrow{AB} + \overrightarrow{BD}\right) = a + (b+c) \quad \textbf{(10)}$$

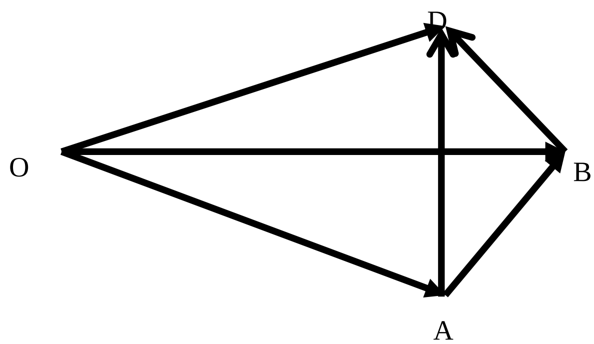

Fig. (7). Geometry of commutative and associative laws.

5. MULTIPLICATION AND DIVISION OF A VECTOR BY SCALAR

Multiplication of a vector by scalar can be defined as [61]:

$$|na| = |n| \; |a| \quad \textbf{(11)}$$

To obtain unit vector, we divide the vector by its magnitude as follows:

$$\hat{A} = \frac{\vec{A}}{|A|} \quad \textbf{(12)}$$

6. DIVISION OF A SEGMENT IN A GIVEN RATIO

To divide any segment in a given ration, let us assume that we have two position vectors a and b, as shown in Fig. (**7**), then it will be required to find another position vector r that will divides the straight line AB internally by the ration $m:n$

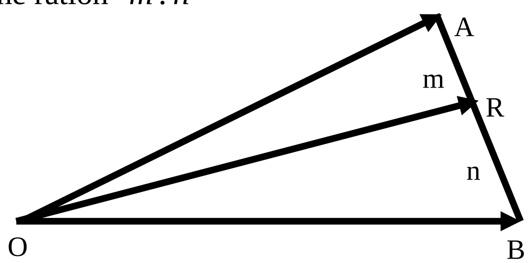

Fig. (8). Division of a segment in a given ratio.

Referring to Fig. (**8**), from the triangle OBR, one can write:

$$\overrightarrow{OB} = \overrightarrow{OR} + \overrightarrow{RB}$$

Or

$$\overrightarrow{RB} = \overrightarrow{OB} - \overrightarrow{OR} = b - r \qquad \textbf{(13)}$$

And from the triangle OAR, we have:

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

Or

$$\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = r - a \qquad \textbf{(14)}$$

Then we have:

$$m \cdot \overrightarrow{RB} = n \cdot \overrightarrow{AR} \qquad \textbf{(15)}$$

Making use of equations (13) and (14) into (15), one can get:

$$m \cdot (b - r) = n \cdot (r - a)$$

Therefore;

$$r = \frac{mb+na}{m+n}, m+n \neq 0 \qquad \textbf{(16)}$$

7. COMPONENTS OF VECTORS

In Cartesian coordinate system xyz, let the initial point be $P(x_1, y_1, z_1)$ and the terminal point is $Q(x_2, y_2, z_2)$, therefore any vector between these two points will have three components as follows [62]:

$$\begin{aligned} a_1 &= x_2 - x_2 \\ a_2 &= y_2 - y_2 \\ a_3 &= z_2 - z_2 \end{aligned} \qquad \textbf{(17)}$$

These components can take the other following form:

$$a = [a_1, a_2, a_3] \qquad \textbf{(18)}$$

In terms of these components, the magnitude of the vector will be:

$$|a| = \sqrt{a_1^{\ 2} + a_2^{\ 2} + a_3^{\ 2}} \qquad \textbf{(19)}$$

8. ALGEBRAIC VECTOR ADDITION

Assume that we have two vectors, written in terms of their components $a[a_1, a_2, a_3]$ and $b[b_1, b_2, b_3]$, therefore:

$$a + b = [a_1 + b_1, a_2 + b_2, a_3 + b_3] \qquad \textbf{(20)}$$

9. BASIC PROPERTIES OF VECTOR ADDITION

Assume that we have three vectors a, b, c, then the following properties hold:

$$a + b = b + a \quad \textbf{(21)}$$

$$(a + b) + c = a + (b + c) \quad \textbf{(22)}$$

$$a + 0 = 0 + a = a \quad \textbf{(23)}$$

$$a + (-a) = 0 \quad \textbf{(24)}$$

10. INNER (DOT) PRODUCT

10.1. Inner Product in Terms of the Angle

The dot product of two vectors is denoted by $(.)$, therefore if we have two vectors a, b, then the dot product is given as [63]:

$$a.b = |a||b| \cos \gamma \quad \textbf{(25)}$$

Where γ is the angle between the two vectors.

Example

Find the dot product for the following two vectors, $\vec{a}$ and $\vec{b}$ and the angle between them is 30 degree, given that their moduli are 6 and 9 respectively.

Solution

The given data can be summarized as follow:

1- Two vectors $\vec{a}$ and $\vec{b}$

2- The angle between the two vectors is 30

3- The magnitude of the two vectors are 6 and 9

Then:

$$a.b = |a||b|\cos\gamma = 6 \times 9 \times \cos 30$$
$$= 27\sqrt{3}$$

10.2. Inner Product in Terms of their Components

Assume that the two vectors are given in terms of their components, $a[a_1, a_2, a_3]$ and $b[b_1, b_2, b_3]$, then the inner product can be written as:

$$a.b = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{26}$$

11. VECTOR (CROSS) PRODUCT

The cross product is completely differs from inner product starting from its mathematical symbol up to the final result. In the inner product leads to a scalar quantity while the vector product leads to another vector perpendicular to the plane containing the two vectors [64]. Assume that we have two vectors given in terms of their components $a[a_1, a_2, a_3]$ and $b[b_1, b_2, b_3]$, then:

$$c = a \times b \tag{27}$$

Also

$$a \times b = |a||b|\sin\gamma \tag{28}$$

And finally;

$$c = a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{29}$$

12. VECTOR AND SCALAR FUNCTIONS

Vector calculus involves two types of functions, vector and scalar function [65].

12.1. Vector Function

It is the function whose values are ***vectors***; *i.e*,

$$v = v(p) = [v_1(p), v_2(p), v_3(p)] \quad \mathbf{(30)}$$

depending on the points p in space. In applications, the domain of definition for such a function is a region of space or surface in space or a curve in space.

We say that a vector function defines a vector field in that region. Examples are shown below in the following figure. In Fig. (**9**) three different examples of vector function, the field of tangent vectors of a curve, the field of normal vector of a surface and finally, the velocity of a rotating body [66].

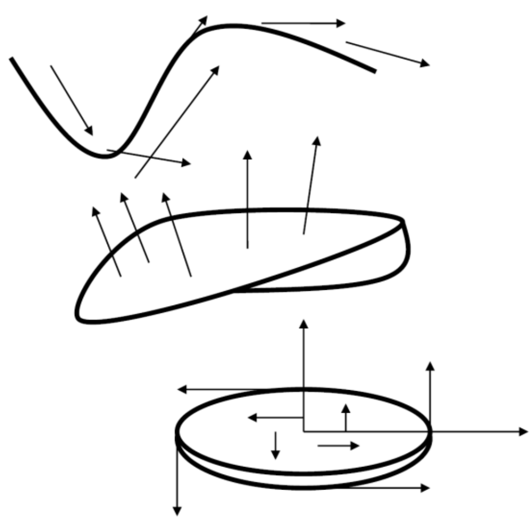

Fig. (9). Vector function.

12.2. Scalar Function

It is the function whose values are ***scalars***; i.e,

$$f = f(p) \tag{31}$$

depending on the points *p* in space.

A brief example of the scalar function is the distance $f(p)$ of any point p from a fixed point p_0 in space is a scalar function whose domain of definition is the whole space.

Example

f(*p*) Defines a scalar field in space. If we introduce a Cartesian coordinate system and p_0 has the coordinates (x_0, y_0, z_0), then f is given by the well-known formula:

$$f(p) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \tag{32}$$

An important note is that the direction cosines of the line through p *and* p_0 *are not scalar function because their values will depend on the choice of the coordinate system.*

Example

At any instant the velocity vectors *v*(*p*) of a rotating body constitute a vector field, the so-called velocity field of the rotation. If we introduce a Cartesian coordinate system having the origin on the axis of rotation, then:

$$\begin{aligned} v(x, y, z) &= w \times r \\ &= w \times [x, y, z] \\ &= w \times [xi + yj + zk] \end{aligned} \tag{33}$$

Where (x, y, z) are the coordinates of a point p at the instant under consideration. If the coordinates are such that the z-axis is the axis of

CHAPTER 7

Vector Differential and Integral Calculus

Abstract: The topics vector differential and integral calculus arise in many practical and engineering applications. This chapter concerns mainly with these topics but few revision on the basis of vectors is introduced. Scalar and vector functions are also introduced due to their importance in numerical analysis.

Keywords: Inner product of vectors, scalar function, vector definition, vector product, vector function.

1. VECTORS QUANTITIES

A vector describes a physical quantity, such as displacement velocity and acceleration. It is represented by a bold letter **V**, or V̲ We will describe in this chapter operations like, addition subtraction, multiplication of vector (dot and cross products). A *direction* and a *magnitude* describes a vector. Fig. (**1**) shows $\vec{A}\&\vec{B}$ are two vectors with different magnitude and direction.

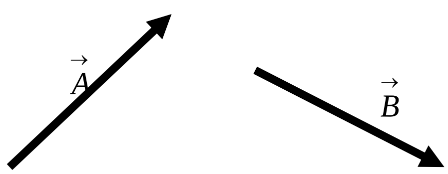

Fig. (1). $\vec{A}\&\vec{B}$ are two vectors with different magnitude and direction.

1.1. Vector Definition

A vector is a quantity that is determined by both its magnitude and direction. The vector may describes a physical quantity, such as displacement velocity and acceleration [58].

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1.2. Vector Length

The length (Magnitude) of the vector a is also called the norm (or Euclidean norm) of the vector a and is denoted BY $|a|$.

1.3. Equality of Vectors

Two vectors $\vec{A} \& \vec{B}$ are said to be equal if they have the same magnitude and direction.

2. COMPONENTS OF UNIT VECTOR

Any vector has a three components associated with the principle axes as follow [59]:

$$i = [1,0,0] \qquad j = [0,1,0] \qquad k = [0,0,1] \tag{1}$$

These unit vectors are shown in Fig. (**2**), with coordinates on the axes.

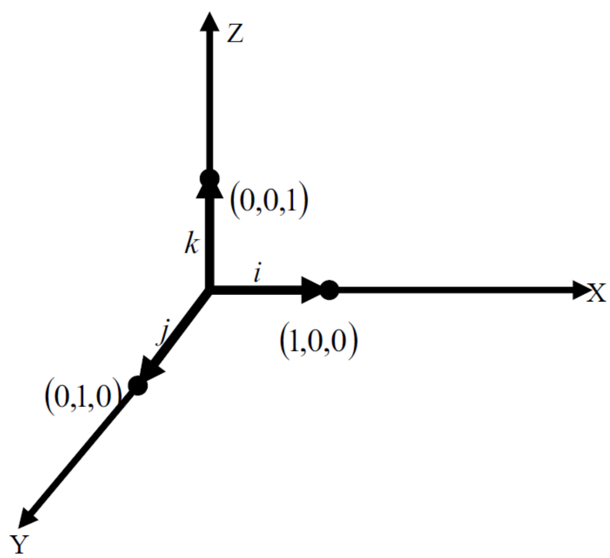

Fig. (2). Unit vector and its components.

3. SPECIAL TYPES OF VECTORS

3.1. Zero – Null – Vector

It is a vector having no direction and zero magnitude and denoted by the symbol O.

3.2. Proper Vector

The proper vector is defined as any vector of any magnitude differs from zero and have defined direction

3.3. Position Vector

A Cartesian coordinate system being given, the position vector r of a point $A:(x, y, z)$ is the vector with the origin $(0,0,0)$ as the initial point and $A:(x, y, z)$ as the terminal point. Thus $r:[x, y, z]$

4. GEOMETRICAL ADDITION AND SUBTRACTION OF VECTORS

To deal with vectors, it is very important to be in a good understanding well the geometric interpretation of the vector, and then we can move after that to addition, subtraction, and multiplication [60]. Let us refer to Fig. (**3**), and assume that we have two vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$

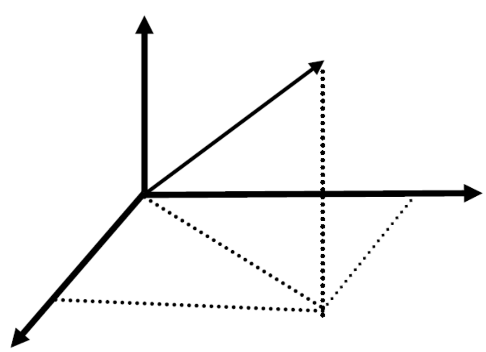

Fig. (3). The position vector and its components.

CHAPTER 8

Special Functions

Abstract: Special functions are mostly wide and found in different branches of practical and engineering problems. No one can cover all different types and classifications of such functions. In the present chapter, we will do our best to cover most and famous special functions. This study will cover main topics in each type starting from the basic definition, mathematical formula, properties and more.

Keywords: Bessel functions, beta function, error function, Gamma function, Legendre polynomials, orthogonal property, Rodrige polynomials.

1. INTRODUCTION

The ***Gamma*** and ***Beta*** functions are two important special functions. The special function is defined as an integral or solution of differential equation. Defined as integral as in the case of Gamma and Beta functions, and defined as the solution of a differential equation as in the case of ***Bessel*** functions, ***Legendre*** and ***Rodrige*** polynomials.

2. GAMMA FUNCTION

Gamma function is one of the most important functions that simplifies evaluating integrals, which indeed, difficult to evaluate it analytically. Gamma function, mathematically, defined as [74]:

$$\Gamma(n) = \int_{0}^{\infty} x^{n-1} e^{-x} dx \qquad \textbf{(1)}$$

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If one integrate by parts, yields:

$$I = \left\{ x^{n-1}\left(\frac{e^{-x}}{-1}\right)\right\}_{x=0}^{x=\infty} + (n-1)\int_0^\infty x^{n-2}e^{-x}dx \quad \textbf{(2)}$$

In equation (2), the first term vanishes at both limits of integration, therefore, it becomes:

$$I = (n-1)\int_0^\infty x^{n-2}e^{-x}dx$$
$$= \Gamma(n-1) \quad \textbf{(3)}$$

Therefore, the gamma function takes the new form:

$$\Gamma(n) = (n-1)\Gamma(n-1) \quad \textbf{(4)}$$

From equation (4), it is clear that the gamma function is known throughout a unit interval, say; $1 < n \leq 2$, then the values throughout the next interval are found, subsequently the next one and so on. The graph of gamma function is shown in Fig. (**1**).

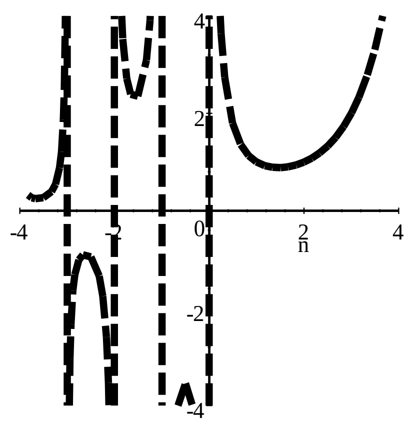

Fig. (1). Graph of gamma function.

3. VALUES OF GAMMA FUNCTION

Values of gamma function differ according to the value of the variable *n*. This variable may take, positive, negative, fraction and or integer. Next, we will show how gamma function will be according to the corresponding value.

3.1. Case (1) Positive Integer

In this case, the gamma function takes the following formula:

$$\Gamma(n) = (n-1)! \qquad \textbf{(5)}$$

Example

$$\Gamma(5) = (5-1)! = (4)! = 24 \qquad \text{(Ex1-1)}$$

3.2. Case (2) Positive or Negative Fraction

In this case, two different formulas are used according to the sign of the variable n positive or negative.

3.2.1. Case (2-1) Positive Fraction

In this case, the gamma for a positive fraction evaluated according to the following formula:

$$\Gamma(n) = (n-1)\Gamma(n-1) \qquad \textbf{(6)}$$

It is important to remember that we apply equation (6) to the fraction n till the value of n becomes in the interval $1.1 \le n \le 1.9$*, then make use of Table **1**.*

Table (1). Tabulated values of Gamma function.

n	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
$\Gamma(n)$	.951	.918	.898	.887	.886	.895	.905	.931	.962

CHAPTER 9

Real and Complex Fourier Series

Abstract: Fourier series is one of the topics that used in function approximation, especially when it is recommended using series expansion instead of using the function it-self. But one may ask himself just simple question, is the Fourier series can be good alternative than the original function?. The answer is actually no and no one can say we can get accurate approximation for the function but we can say Fourier series is quite approximation for the function within specified period under prescribed conditions related to the function itself.

Keywords: Complex Fourier Expansion. Fourier series, Half-Range Expansion, Odd and Even Function,

1. INTRODUCTION

Fourier series are series of cosine and sine terms and arise in the important practical task of representing general periodic functions. The theory of Fourier series is rather complicated, but the application of these series is simple. These series, named after the French physicist JOSEPH FOURIER (1768-1830) [91], are a very powerful tool in connection with various problems involving ordinary differential equations. In the present chapter, we shall discuss the basic concepts, facts and techniques in connection with Fourier series.

2. PERIODIC FUNCTIONS

A function $f(x)$ is said to be periodic if and only if there exist a positive

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number T such that for any x in the domain, $f(x+T)=f(x)$ where T is called the Period of $f(x)$, the graph of the function is shown in Fig. (**1**) [92].

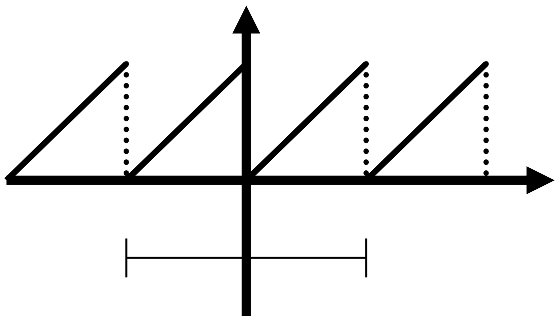

Fig. (1). Saw tooth periodic function.

Example (1)

Find the period for the following function, $f(x)=\cos x$

Solution

To find the period of the function $f(x)=\cos x$, we apply the concept of the periodic functions.

$$f(x)=f(x+T)\Rightarrow \cos x=\cos(x+T)$$

$$\cos x=\cos x\cos T-\sin x\sin T$$

To achieve this equation, the first term in the right hand side should reduce to $\cos x$ while the second term should equal to zero. Therefore, $\cos T=1\Rightarrow T=2\pi$

Example (2)

Find the period for the following function, $f(x)=\sin x$

Solution

To find the period of the function $f(x)=\sin x$, we apply the concept of the periodic functions.

$$\sin x = \sin x \cos T - \cos x \sin T$$

To achieve this equation, the first term in the right hand side should reduce to cos x while the second term should equal to zero. Therefore, $\cos T = 1 \Rightarrow T = 2\pi$

Example (3)

Graph the following periodic functions:

$$f(x) = \begin{cases} -k & -1 < x < 0 \\ +k & 0 < x < 1 \end{cases}$$

With period $T = 2$

Solution

The graph of the given function is shown in the Fig. (**2**) below, taking into consideration that the periodic function repeats its-self each period.

3. FOURIER SERIES

Fourier series arise from the practical task of representing a given periodic function in terms of sine and cosine functions. A periodic function $f(x)$ with period $T = 2\ell$ can be represented by a trigonometric series, called Fourier series as follows [93]:

$$f(x) = a_o + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right) \quad \textbf{(1)}$$

The coefficients that appear in the above expansion are called Euler coefficients, which are unknown.

The first coefficient a_o can be determined according to the following procedure.

Determination of a_o

Integrate both sides of the above series, as follows:

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S. G. AHMED

S. G. Ahmed was born in 1962 in Sharkia, Egypt. He is since 2005 Professor of Applied and Engineering Computational Mathematics and Numerical Analysis, Department of Engineering Mathematics and Mathematics, Zagazig University, Zagazig, Egypt. He was previously employed since 1985 at the same department and college. He received his Ph.D. in Applied and Engineering Computational Mathematics and Numerical Analysis from the University of Portsmouth University and Zagazig University through channel system between UK and Egypt in 1996. He worked as a director of Computational Mechanical Center at Zagazig University, between 2007 and 2010. His present research interest is computational Engineering Mathematics and Numerical Analysis related to different branches of Engineering. He published 51 scientific papers and 8-books in English and Arabic languages. He contributed to more than 50 technical reports. He presented introductory invited talks at more than 10 international conferences. He received the following awards and recognitions: 2006-2016 Who is Who in the world, from USA, EMERALD Literati Highly commended author award, 2006, Symbol of Excellency from Cambridge University, U.K., 2011, and ABI: Scientific Award of Excellency, USA, 2011. He is currently a member of the editorial boards of different international journals with impact factor and active member for different international conferences.