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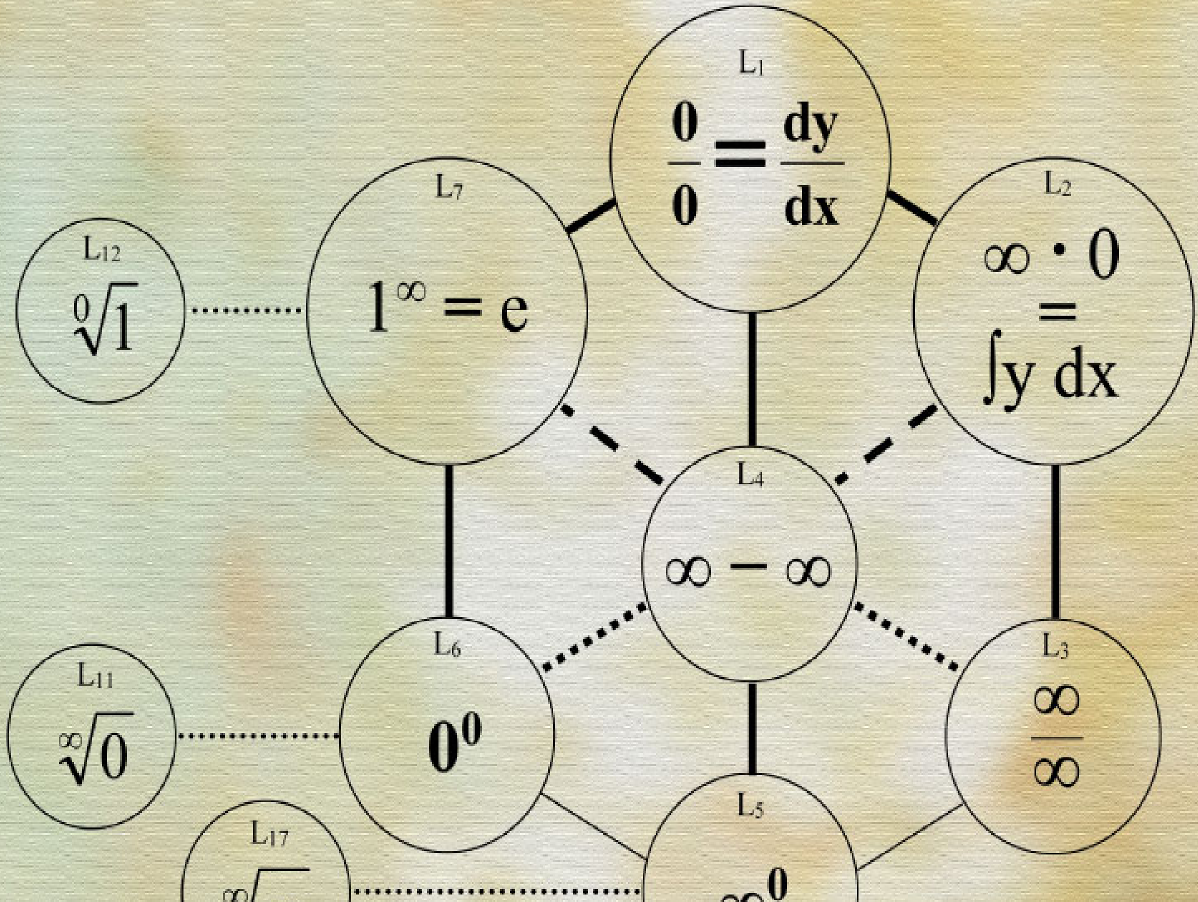
CALCULUS: THE LOGICAL EXTENSION OF ARITHMETIC



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Calculus:

The Logical Extension of

Arithmetic

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FOREWORD

The main objective of this book is to describe a novel way of teaching elementary single-variable calculus that emphasizes an algebraically based formulation of the usual concepts of differentiation and integration. Actually, the novelty lies mainly in the emphasis and interpretation of the time-honored fundamentals of the calculus, not really in the underlying details. A central point of the pedagogical approach, which covers the usual topics treated in an introductory one-semester calculus course, is that such things as division by zero and multiplication by infinity need not be avoided – and are even useful - as long as they are properly defined.

After an introduction in Chapter 1 in which Dr. Elk outlines his pedagogical philosophy for teaching calculus, this treatise begins in Chapter 2 with a fairly traditional introduction to basic concepts such as functions, algebraic equations, sums, products, and inverses. It then deals with coordinate systems, emphasizing that the choices are far richer than merely Cartesian. Dr. Elk stresses that division by zero is acceptable if the "number" infinity is introduced and 0/0 is interpreted correctly. Also included are a description of the fundamentals of plane trigonometry, polar coordinates, a brief introduction to spherical trigonometry, which is rare for elementary calculus books, and the definition of logarithms as inverses of exponentials.

Chapter 3 is focused on differentiation, which is identified with the algebraic expression 0/0, if properly interpreted. This interpretation is just the usual (ε-δ) limit of a difference quotient. The main novelty in the treatment of limits is that l'Hospital's rules are imposed *a priori*, and are used to handle what are called "l'Hospital indeterminate forms", which include 0/0, ∞/∞, and 1^∞. In keeping with this theme, Dr. Elk defines algebraic properties of ∞ that are more or less the same as done for the extended real numbers in more advanced analysis texts. After defining the derivative in terms of the limit quotient, the representation of derivatives in terms of ratios of differentials in Leibniz's notation is treated algebraically and used to verify such properties as the chain rule in a manner similar to that used in the infinitesimal approach to calculus introduced by Abraham Robinson.

In Chapter 4, Dr. Elk introduces integrals interpreted as $\infty\,0$ defined in a "constrained way", which is essentially just the usual limit of Riemann sums. This is followed by a description of some of the basic methods of integration such as substitution and integration by parts. An interesting and unusual aspect of the approach is that the fundamental theorem of integral calculus is taken as an intuitive property of integration. The author covers some topics in this chapter that are rarely seen in elementary texts such as the Dirac delta distribution ("function") as the (distributional) derivative of the Heaviside step function, and a brief intuitive description of Lebesgue integration. Chapter 5 contains a fairly traditional treatment of analytic geometry with a focus on quadratic curves and surfaces.

In Chapter 6, Dr. Elk uses the indeterminate 1^{∞} as a focal point for making connections among the base of the natural logarithm, and integral representation of natural logarithms and l'Hospital's rule. These themes are nicely interwoven and used as a source of logarithmic differentiation. The chapter ends with a very clear treatment of hyperbolic functions that stresses analogies with the trigonometric functions.

Infinite sequences and series are covered in Chapter 7. One novelty here is that the author delves into uncountable sequences, which leads to further consideration – albeit brief – of the concepts of cardinal and ordinal numbers, culminating with an intuitive description and comparison of the cardinality of the integers and the real line. The standard material on convergence tests, comparison tests and power series is covered as well.

In summary, Dr. Elk has presented a clear, lively and thought-provoking alternative to the traditional way of teaching elementary calculus, which might prove to be a viable replacement for current practice. Only time will tell if Dr. Elk's vision resonates with mathematical educators and students.

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PREFACE

To say that the understanding and teaching of calculus has not changed much since its formulation by Liebnitz and Newton, over three centuries ago, may be overkill. **Then, again, it may not be!** Upon reviewing all of the calculus books on the shelf in any large university library, the astute observer will become aware just how staid the associated pedagogy actually is! Not only will they conclude that, the subject matter which was developed in the late seventeenth/early eighteenth centuries still loudly resonates today, but also that the focus of the respective past and present authors **has always been related to <u>HOW</u>, in contradistinction to <u>WHY</u>**. Except for the inclusion of computer graphics, which assists in visualizing selected applications, there has been a scarcity of new ideas that would assist comprehending the foundation upon which the underlying mathematics has been built. **<u>IFF</u>**[1] the subject matter was readily understood, this would not be a liability. **TO THE CONTRARY**, most otherwise educated persons consider the very word "calculus" as being synonymous with "way beyond comprehension".

The perspective that is herewith being promoted is predicated on showing that the understanding and use of calculus should be no more difficult than was the mastering of the earlier studied subsets of mathematics, namely arithmetic and high school level introductions to algebra and geometry. In this treatise it shall be demonstrated that, **when properly presented**, the fundamentals of calculus are the very same ideas that were presented in those earlier more intuitive, and consequently deemed to be more elementary, studies. The only major difference is that they are now being adjoined with a single, supposedly already familiar, concept: namely, that there is no last number in the counting scheme ─ an idea that will be re-enforced in the succeeding pages. In the same manner as the recognition that a number called "zero" was a useful concept when it was postulated a little over a millennium ago, a similar useful number called "infinity" underlies the foundation for the development of a comparable extension of the number system. In fact, these two numbers plus the most primitive of all numbers, one, forms a set whose various binary combinations extend arithmetic into the domain of "calculus".

At this point, in order to counter the prevalent mindset that has been associated with calculus ever since its earliest development, the reader's attention is directed to one of the many muses who influenced the writing of this monograph: An important fundamental classic of mathematics literature, that nearly all educated mathematicians, along with most physical and biological scientists and engineers of all fields, as well as economists and social historians, and **anyone** interested in understanding either the evolution or the implementation of mathematical thought is a monograph by Richard Courant and Herbert Robbins entitled <u>What</u>

is Mathematics? Excerpting from that treatise, which was first published in 1941 and is still in print and in wide circulation nearly three-quarters of a century later, one encounters the following passages which describes precisely what this author hopes to accomplish in his limited approach to the understanding of that very important, but sadly misunderstood, branch of mathematics referred to as "calculus".

> "For more than two thousand years some familiarity with mathematics has been regarded as an indispensible part of the intellectual equipment of every cultured person. . . . Teachers, students, and the educated public demand constructive reform . . . It is possible to proceed on a straight road from the very elements to vantage points from which the substance and driving forces of modern mathematics can be surveyed. . . . The present book is an attempt in this direction. . . . It requires a certain degree of intellectual maturity and a willingness to do some thinking on one's own. This book is written for beginners and scholars, for students and teachers, for philosophers and engineers, for class rooms and libraries."

Our approach begins by noting that calculus has been, **and continues to be,** taught as though it was a whole new subdivision of mathematics, with a unique protocol that required the concomitant "mastering" of rules associated with a new discipline. Instead of pursuing such an ossified path by solving a given set of arcane problems using an even more arcane set of previously developed "algorithms" (*i.e.*, memorization), this author follows a protocol which historically had been deliberately rejected; namely, the domain of calculus is viewed as an extension of what has come to be called "elementary algebra"[2,3,4]. This is equivalent to asserting that the ***unthinking*** application of many of the rules established in traditional arithmetic/algebra courses need to be rejected. Such a proposition is especially true when the limits on these rules are breached. In particular, we focus our attention on the eight possible binary combinations of three "tripartite numbers" (0, 1 and ∞)[5] as they impact the six fundamental operations of arithmetic (addition, subtraction, multiplication, division, raising to a power, and extracting a root). In the process we shall discover that our proposed term "tripartite number"bears an important relationship to the philosophical ideal of "none", "some", and "all", along with its place in symbolic logic and in probability/statistics. Furthermore, we shall conclude that the (so named) 'Fundamental Theorem of Calculus' is ***not-so-fundamental***, or, more accurately, that it is insignificant; ***not even worth the appellation of a "corollary", no less a "theorem"***! One would not be amiss in concluding that the philosophy espoused comes directly from the Gilbert and Sullivan operetta H.M.S. Pinafore: "Never mind the Why and Wherefore!"

In the evolution of our perspective, we probe both the denotation and the connotation of that mathematically definable 'term' **number**, along with three related concepts of importance:

1. extension from the primitive idea of counting;
2. the ideal of repetitive performance of an arithmetic operation
   and
3. the ideal of "inverses"; *i.e.*, the undoing of an arithmetic operation;

In practice, one notes that addition is merely a means of counting from a different starting point than from the heuristic concept of "none". Additionally, in retrospect, one observes that it was not until a little more than a millennium ago, that the foundation of mathematics was shifted from its emphasis on what we today relegate to the subdomain of "geometry" to a new paradigm with the independent postulation of positional notation by the Hindu and the Arabic civilizations, thereby replacing the then prevalent concept of an additive numbering protocol. With the elevation of that amorphous concept of "none" as a "number", denoted by "zero", this new paradigm accentuated the formalization of counting in the reversed direction; *i.e.*, "subtraction". This was then followed by repetitive operations of addition and subtraction, which were designated as "multiplication" and "division" respectively and by a second repetition of these two processes to yield "exponentiation" (alternately referred to as "raising to a power") and root extraction. This set of six "elementary" operations is the foundation, not only for "numbers", but also for many of the more sophisticated concepts in mathematics, which comprise advanced domains of both understanding and of manipulation, calculus being one of them.

At this time, one further expands their purview from the intuitive concept of *none* as the number *zero* to the intuitive concept of *all* as another mathematical term, loosely called "infinity" in the common domain, which shall be more accurately defined later in this text; and thirdly to the **word "**some", with its implication of "one or more". To redirect this latter word into being a term we delimit its scope so that it has the denotation of precisely 1 in the following development of the foundations of calculus. That this can be done without loss of generality is illustrated in Section 2.2. where one of the arbitrary choices in the process of establishing a measuring system is the choice of a unit reference. The judicious selection of such a reference simplifies the process of computation..

Our entire new premise (hopefully a new paradigm) as to "WHAT IS CALCULUS?" begins by focusing attention on the fifty-four permutations of the eight possible binary combinations of the above delineated six fundamental operations of arithmetic. This is then followed by the recognition that such a protocol gives rise not only to the seven traditional "l'Hospital indeterminate forms"[6], but to a larger set of similarly-related indeterminate forms. Such a development is promoted to center stage early in Chapter 3. Only a selected re-examination[7] of pre-calculus material[8] in Chapter 2, and on the concepts of limits and continuity in the first section in Chapter 3 has been given priority over what we interpret as the fundamental subject matter of calculus.

We further assert that the concept of infinity is an important number. **YES! NUMBER!** (even though, to some, it may have "unfavorable" social or political associations). At this point one is reminded of the begrudging acceptance, over a millennium ago, of a mathematical term to denote "nothing". It was that introduction, **and the acceptance**, of zero **as such a number** which caused the paradigm shift which facilitated mathematics entering the public psyche. It is important to be aware that in the previous prevailing counting systems, exemplified by Roman numerals, there was (and remains today) no symbol for nothing. To the contrary, numbers were "naturally" limited to the set of positive integers, which were designated as "natural numbers". With the establishment of zero as a number, arithmetic entered the domain of the masses, rather than being confined to an esoteric pursuit of a, mostly religious, elite. Additionally[9], it fostered new developments in algebra and analysis.

We herewith sponsor the introduction of a similar perspective concerning infinity, examining its influence on mathematics in general, and on calculus in particular. This is done by focusing attention on a so-called "Eleventh Commandment":

> Thou shalt not divide by zero!

This, admittedly irreverent, term which had been satirically proposed by some unrenowned anti-clerical mathematician, spawned the even more infamous Twelfth Commandment. A treatise "Hegel, Haeckel, Kossuth and the Twelfth Commandment" satirized one earlier and two contemporary "would be intellectuals". That treatise[10] avowed that ever since Moses brought the Ten Commandments from Mount Sinai, society has sought an eleventh. Whether there is such a commandment is speculation; **HOWEVER**, *just in case there does exist such an eleventh commandment*, here is a twelfth.[11,12] It **should** be regarded equally as important as the original ten:

> Thou shalt not set thyself up as an expert in a field thou doeth not understand!

Chowolson disdained anyone venturing too far afield without the prescribed, authenticated knowledge background. He avowed a "genius" in one field [13] might well be an "idiot" in another. Moreover, even articles published in highly prestigious journals do not guarantee erudition.[14]

The problem with such a premise, however, is its unstated assumption: that the previous "geniuses", those who established the paradigms that we believe today, are infallible. Were this to be true, all of mankind would have reached the end of discovering new knowledge and all progress would be impossible.[15] To the contrary, there is no indelible line that should NOT be crossed. We often need to suffer "innovators" (translation = fools) graciously.

In our present treatise, we knowingly accept the accusation of non-conformity. However, just as truth refutes liable, we dismiss the premise that only recognized authorities have the prerogative of advancing new ideas; *i.e.*, that all others stand accused of heresy and of violating the Twelfth Commandment.

To the contrary, we challenge the historical method of teaching calculus. We place our emphasis on a re-examination of what are called "indeterminate" forms. Such forms emanate from performing algebraic manipulations "in the wrong place" or "at the wrong time". We further assert NEVER say "Thou shalt NOT ....." to anything in mathematics or science. This is in contradistinction to **religion —** which is based on faith. We espouse probing the consequences that result from violations of traditional protocol. We further champion Lakatos's technique[16] and examine how violations can be evaded. In our approach, meaning is given **under the appropriate conditions** to otherwise indeterminate forms: For example, division by zero**, when performed blindly,** leads to ridiculous statements. However, with the appropriate restraints, such division leads to calculus and transfinite numbers. Each of these different protocols adds substantially to both our communal and our individual understanding of logic, mathematics and science.

Similarly, we re-examine the role played by infinity in the developing of mathematical processes. The inverse operation of "anti-differentiation" is envisioned as adding meaning to "infinity multiplied by zero". A third and fourth of these indeterminate forms (infinity minus infinity and one raised to the infinite power) underlie a system of "natural" logarithms. The base of this logarithm system is a fundamental constant designated by the letter (**e**).

Moreover, when these conditions are not met, mathematical chaos results. The worst fears, expressed by the computer slang term GIGO (**Garbage In** yields **Garbage Out**) are encountered. This includes familiar nonsense such as a "proof" that: 2 = 1. We shall return to this quote-proof-unquote in Chapter 2, Section 3. Were such an equation to be true, all of mathematics would be useless!

Notwithstanding what many traditionalists will undoubtedly view as the admitted bias of this author, we place a high priority on refuting over-simplified explanations that are fundamentally flawed! In order to advantageously use the principles being here espoused as "calculus", we redevelop, from basic principles, and thus expand, that set of combinations which are known as the "l'Hospital indeterminate forms"[17]. Additionally, in Chapter 3, we shall demonstrate the relationship of these forms to an even more fundamental mathematical foundation.

Reiterating our obsession with over-simplified explanations, we avow that, the desirability of simplicity **SHOULD NEVER BE** at the expense of accuracy. Knowing that the reader will lack the depth of knowledge to be able to raise an objection is insufficient justification for "taking short-cuts". **To the contrary**, an author knowing that his audience is unprepared for a "better" explanation MUST:

a). supply the missing background whenever it is not too advanced
OR
b). acknowledge this limitation and advise what future study is desired.

Two important ideas that permeate this book are:

1. "Half right" answers are **much worse** than wrong answers! This is notwithstanding that such answers can be found in many, often highly regarded, textbooks, as well as, unfortunately, other mathematics and science publications. Intellectual honesty **demands** that when better explanations are available they **need** to be acknowledged. A problem area should never be "swept under the rug". We disdain placing blinders on the advanced student and feel honor-bound to discuss (a) "what is the source of the problem?' and (b) 'what are our limitations?'
2. As well as the "algebraic logic" which underlies calculus, this text stresses "geometric logic". For example, consider the 'slices' *vs.* 'shells' method of determining volumes of revolution. We object to the traditional "Charge of the Light Brigade" mentality:

   Their's not to make reply.
   Their's not to reason why.
   Their's but to do, and to die.

Instead of just dumping the traditional integration formulas onto the student, the discussion of this technique is framed as part of a two parameter system. Now, by differentiating the formula for the volume of a cylinder

$$V = \pi r^2 h$$

by each of its independent variables r and h, we obtain the resultant formulas referred to as the "method of 'shells' *vs.* 'slices'". In other words, this pedagogic "trick" simplifies the remembering and correct usage of the appropriate formulas, which is especially useful when applied to "moving" an axis of rotation.

Of far more geometric importance is a major mistreatment common in vector arithmetic. Our perception "What a Vector is" is ***CONTRARY TO WHAT MOST CALCULUS TEXTBOOKS TEACH***. Because we never want the advanced student to have to "unlearn" mathematically wrong ideas, we vociferously object to regarding the cross product as **just another vector**. This would be the scenario for those continuing advanced studies in electrodynamics if the traditional description of a vector as "a quantity having magnitude and direction" was employed. Such a misconception in the late 1800s helped to reinforce belief in the ether. We avoid this physics inaccuracy by giving a more accurate definition for that quantity called a "vector". Because a physically **accurate** presentation **MUST** allow for all three metric motions: translation, rotation AND reflection of the coordinate axes, we examine some properties of a more complex form of mathematical structure called a "dyadic" and introduce a more advanced (third) type of vector multiplication. This is then combined with a geometric interpretation of tensor analysis. Only then are we able to fulfill our stated goal which is to present a mathematically correct presentation of the foundation of vector algebra and calculus, while still maintaining our description at the student's present level of understanding. In [18], this subject matter is developed along with an examination of the problems associated with attempts at formulating a system for vector division. The background material is a series of published research articles by this author[18].

The historical development of another subdomain of calculus, that this author finds undesirable, involves the traditional assigning of names to a set of functions based on circular trigonometry, but then this is inconsistently followed by giving the formal definitions of these functions in terms of exponential functions. Such a protocol to the questioning student will appear to be equivalent to drawing such functions and their associated names out of a hat. This not to say that such a protocol is mathematically wrong, We object precisely because, similar to the above discussion of vectors, there is a familiar, consistent, viable analogy to the trigonometry that was learned earlier. Consequently, we are able to define these new function using a reference hyperbola in the same manner as the similarly named functions were defined using a reference circle; *i.e.*, we define the "hyperbolic trigonometric functions" in terms of their lengths in a reference hyperbola. Then, in a later chapter we show that the correlation to the exponential functions is a derived relation, rather than being the basis for their definition.

Meanwhile, although we shall examine all the traditional properties, our presentation is often unorthodox. For example, consider the algebra "rule" for multiplication by a negative number: Reverse the direction of the original inequality. We treat this **NOT AS A RULE**, but rather as a logical protocol. One merely adds a common term to both sides of an inequality (see Section 2.1). In other words, this result is an application of arithmetic, NOT some memorized rule. Similarly, consider "**the chain rule**", which is traditionally taught in calculus courses. Note this is NOT some "erudition" discovered by an astute mathematician. Neither is it the combined genius promulgated by an august assembly of learned mathematicians. TO

THE CONTRARY, it is merely simple algebra: Multiplying by one in a convenient form. Likewise, in many instances, differentials supply an intuitively simpler technique than derivatives. We recommend using them when performing implicit differentiation. This is in contradistinction to the perspective espoused in traditional textbooks. There, the focus is almost exclusively on derivatives. We, on the other hand regard a derivative as a secondary concept that is merely the division of two differentials (*i.e.*, 0 divided by 0). This must be done, however, UNDER THE APPROPRIATE CONDITIONS. See Chapter 3. Similarly, an integral is EXISTENTIALLY the summation of an infinite number of differentials (*i.e.*, ∞ multiplied by 0). Again this must be done UNDER THE APPROPRIATE CONDITIONS. See Chapter 4.

A further comment of importance is that, when evaluating the rigorous development of **<u>any</u>** idea, the bias of the writer should **<u>always</u>** be taken into consideration. This author regards many developments as being too esoteric for the applied student of science or engineering.[19] We often will take a more mundane approach[20], ***when we believe that it is NOT mathematically wrong***. For example, in Chapter 3, we focus on that order of infinity associated with “countability”.[21] This is unabashedly described as “larger than any number you can count to”. Likewise, in Chapter 7, in the discussion of sequences and series, we introduce that “order of infinity” associated with the continuum[22], as well as including a superficial extension to what are referred to as “the higher orders of infinity”. In a similar manner, we include an “applied” definition of “function” in Section 2.1.[23]. This is in contradistinction to developing one or more of the various “pure” mathematics definitions. Such definitions are more remote from the heuristics that scientists and engineers utilize.

Two personal comments that are herewith interjected are:

1. I have **<u>never once</u>** failed a student in calculus for not knowing calculus. I have, however, failed many students for not knowing algebra (or often arithmetic). Without such background knowledge, the ideas introduced in the calculus class **<u>COULD NOT</u>** make sense. This is in contradistinction to other subjects in which the student may excel. There are no prerequisite courses in the study of a Shakespearean historical play. No knowledge of English history is ***<u>required</u>*** for an essay on Richard III. The author supplied everything you need to know to write your essay. This is NOT to say that such historical knowledge would, or should, go unused. In mathematics, on the other hand, one does not have the ability to do addition without having mastering counting. Likewise, it is extremely unlikely to master subtraction without knowing both addition and counting. This same principle applies to each new level studied. Competence in division requires proficiency in multiplication and subtraction. This cascades to imply the above mentioned competency in addition and counting,

etc. **Demand** for proficiency in algebra plus arithmetic frustrates many calculus students.

2. This proposal is especially, but not exclusively, useful to those teachers of calculus who have a progressive student body **<u>and</u>** an administration willing to allow for experimentation with new ideas. Many years ago, I had just such a combination when I taught geometry to tenth grade students at an elite private school in New York City. My specific "would-be" paradigm shift was to integrate traditional plane and solid geometry into a proposed radical textbook: "Geometry – From 'Gee' Through 'Why'". For example, after developing the properties of triangles (call them "2-simplexes"; *i.e.*, the simplest figures in two dimensional space), my syllabus continued to the corresponding ideas related to tetrahedra (*i.e.*, "3-simplexes") and even dared to superficially venture into hyperspace (*i.e.*, "n-space") BEFORE studying the various important four sided planar figures. This protocol worked magnificently for the top track of an ability segregated student body, especially those who were intellectually mature enough to welcome new ideas; while being a disaster for lower tracks. In retrospect, if my protocol had been applied to the appropriate audience, it most probably would have worked. However, for the environment in which this experiment was done, it was political naivety. The competition **not only** for the initial assignment to the highest possible track, **but more importantly** for being allowed to continue in successive semesters to remain in such a track created an unremitting environment that is akin to a caste system. Although such a protocol could be countermanded by a highly flexible mobility between the different tracks (depending on how the individual students perform in each succeeding module), it was rightly regarded as having toxic consequences. Most important of these was the psychological consequence of the need to move those students who were evaluated as being at the bottom of one track down to the next lower track, in order to make room for those at the top of that lower track to move up. Although this can be easily done in an authoritarian system, where the administration has the power to dictate, through the awarding of grades, and of class assignment, it is much more difficult in a democracy where any value judgments may be viewed as subjective, and where "objective values" are a legalistic fiction that is, at best, arbitrary and capricious. The degree of difficulty in achieving "fairness" (whatever that means) is even more difficult in a private high tuition school where the parents, and not the state, are paying the bill.

On the other hand, such a protocol can achieve its educational objective in a democracy when the attendees are self-selected and where the only presiding judge is the marketplace in which all who can hope to more successfully compete are free to do so. This is precisely where Courant and Robbins' treatise, "What Is Mathematics?", now regarded as an outstanding classic text, earned its place in mathematical literature. We hope to see a similar paradigm shift in the understanding and teaching of calculus. If NOT by us, then let it fall to others who produce a better product.

Notes

[1] IFF is NOT a typographical error, but rather a common abbreviation in mathematics denoting the two propositions IF and ONLY IF simultaneously.
[2] The term[3] "algebra" refers to either one of two subdivisions of mathematics, depending on the context in which the word appears: Traditionally, and most often colloquially, this term, which often is accompanied by the adjective "elementary", denotes the mathematical subject matter that follows arithmetic in the curriculum taught in most junior high schools, At higher levels of education (college and graduate school), on the other hand, with or without the adjective "modern", a domain that includes more advanced topics, such as matrices, tensors, rings, fields, *etc.* is frequently implied.
[3] An important distinction between the nouns "term" *vs.* "word" shall be maintained throughout this treatise: "Term" shall be reserved for a precise denotation by professionals in that field of study. This is in contradistinction to "word" which is in common usage by lay persons and which allows for much less precision in its connotation. Moreover, remember "precision" is a matter of mathematics *vs.* "accuracy" which is dependent on the science. A further important comment, notwithstanding that many may consider it to be "tangential", is the role played by "orismology" [4] (the science of defining words) both in the development of calculus as well as in the earlier evolution of our concepts of arithmetic, especially in the computer age. Remember that a digital computer, unlike the human mind, has as its algorithmic base of computation a much more efficient (BUT SIMULTANEOUSLY MIND-NUMBING) numbering system with only two distinct numbers (zero and one), rather than on the biological 'accident' that the human species has ten digits to count on.
[4]The following passage, which this author published in an earlier treatise "A New Unifying Biparametric Nomenclature that Spans all of Chemistry", Elsevier, 2004, describes both the denotation and the connotation of the term "orismology":

> The term "orismology" has been resurrected from being an arcane synonym of "terminology" to denote a study of the entire evolution of ideas inherent in a term, rather than being limited to the specific connotation of the present usage of the term ─ the usual meaning associated with "terminology".

[5] Note that the term "tripartite number" was chosen over others who have referred to this same set of three numbers, often in a meta-mathematical context. For example, Michael Van Laanen in his dissertation "Encounters with Infinity", Trafford Publishing, 2002 introduced this particular set of three numbers with the term "boundary numbers". We eschew using his term inasmuch as we view the concept of boundary as being associated with other very different denotations, especially in mathematical physics.

[6]Although until recently most textbooks did not introduce the familiar 7 l'Hospital's forms until late in the second semester of a traditional three semester calculus sequence, and even those more recent ones that do at the end of the first semester, they are still regarded as not much more than a glorified footnote.

[7] That **should** have been mastered in preparatory algebra, geometry and trigonometry courses.

[8]Some of which is here presented at a more sophisticated level.

[9]Note the origin of this concept in our everyday language. This is true for "numerous" other mathematics-based terms.

[10]Originally published in 1906 anonymously. Later acknowledged as authored by Professor O. D. Chowolson, Department of Physics, Kaiserlichen University, Saint Petersburg, Russia.

[11] One of my mentors[12] advised this was required reading for German doctoral students in science and mathematics in pre-World War II Germany.

[12] Dr. Julius Friedrich Vandrey, who had been one of the German scientists at Peenemunde during World War II was the Director of Research at The Glenn L. Martin Company in the 1950's. During that time I was a Research Engineer in his department.

[13] The butt of this satire was the biologist Ernst Heinrich Haeckel and philosopher Georg Wilhelm Friedrich Hegel. Haeckel had extrapolated his biological finding to predict a heat sink at the South Pole. Hegel, upon venturing into science avowed:

> "The fixed stars are a pimple on the firmament.
> There can be only 7 planets.
> But that doesn't jibe with the facts!"
> So much the worst for the facts.

This Luddite approach to knowledge in general, and science in particular, caused Karl Marx to comment:

> "I found Hegel standing on his head and set him upright.!"

[14] "Some Observations on Haeckel's Riddle" by H. Kossuth in "Zeitschrift für Philosophie und philosophische Kritik" 1903. This article displayed an appalling lack of understanding of elementary physics. Without Chowolson's treatise, the only claim to fame of H. Kossuth was popular confusion with another Kossuth. Lajos Kossuth was the Hungarian poet who led the 1848 revolution against the Hapsburg monarchy. Historians blame Lajos for the failure of that uprising, due to his arrogance. It is asserted that any decent general, with a modicum of military knowledge, would have won.

[15] A common urban legend attributed to Charles H. Duell, Commissione of the U.S. Office of Patents in 1899 was that "everything that can be invented has been invented". Nowadays, there are many websites about such "bad predictions" such as http//www.rinkworkks.com. These include, among other things, the comment of Thomas

Watson, president of IBM, that there would be a market for at most 5 computers in the entire world.

[16] Imre Lakotas in his 1976 treatise: "Proofs and Refutations — The Logic of Mathematical Discovery" Cambridge University Press.

[17]http://en.wikipedia.org/wiki/Johann_Bernouilli. Bernoulli was hired by Guillaume de L'Hôpital for tutoring in mathematics. Bernoulli and L'Hôpital signed a contract which gave l'Hôpital the right to use Bernoulli's discoveries as he pleased. L'Hôpital authored the first textbook on infinitesimal calculus, *Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes* in 1696, which mainly consisted of the work of Bernoulli, including what is now known as L'Hôpital's rule.

[18] International Journal of Mathematics Education in Science and Technology: (a) "Refining the Definition of Indeterminate Forms", 30(6),1999,924-928; (b) "Is Calculus Really that Different from Algebra? A More Logical Way to Understand and Teach Calculus", 29(3), 1998, 351-358; (c) "The Cross Product of Two Vectors is NOT Just Another Vector - A Major Misconception Being Perpetuated in Calculus and Vector Analysis Textbooks", 28(4), 1997, 531-543.

[19]This decision is purely a heuristic one.

[20]Some may say naïve.

[21]Denoted as $\aleph_0$ in the development of transfinite numbers by Georg Cantor

[22]Cantor's $\aleph_1$, also often denoted by c

[23]The International Dictionary of Applied Mathematics, Van Nostrand, Princeton, 1960, p.386.

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CONFLICT OF INTEREST

The author confirms that author has no conflict of interest to declare for this publication.

DEDICATED TO

My Wife:	Rosalind Elk
My Daughters:	Marilyn Elk Jacob & Janet Elk Davis
My Grandchildren:	Andrew Jacob; Jaimee Davis; Rebecca Jacob and Melissa Davis

CHAPTER 1

Calculus as the Logical Extension of Arithmetic, rather than as a Whole New Subset of Mathematics

Abstract: In order to understand not only "how", but also "why", calculus evolved into that major subdivision of mathematics that it presently occupies, the binary arithmetic of three special numbers is selected as the starting point. When these three, herewith designated by the adjective "foundational", numbers are combined with the six familiar arithmetic operations of addition, subtraction, multiplication, division, raising to a power and extracting a root, the basis for a new, enlarged perspective of 'what is mathematics?' is created. Note that the selected term, foundational number, was chosen over others who have referred to this same set of three numbers in a meta-mathematical context. Notwithstanding that this same set of three numbers was designated in [1] as "boundary numbers", such a name is herewith eschewed as the traditional concept of "boundary" has a very different denotation in mathematical physics.

Rather than rigidly following the traditional approach, based on function theory, which has been taught in high schools and colleges ever since it was independently developed by Leibnitz and Newton a third of a millennium ago, this treatise focuses attention on the arithmetic taught in elementary school, but from a more advanced standpoint in much the same manner as Felix Klein's Erlanger Program and its Elementary Mathematics From an Advanced Standpoint [2]. The single-most important difference in this formulation is that there is a new perspective which incorporates that long familiar, but not fully exploited, recognition that there does not exist a last number in the counting sequence. In other words, for any given number, call it "n", there is another unique number n+1. Similarly, for n+1 there is an n+2; *etc*.

Because of the above, instead of the traditional side-stepping of the question, both this author and the reader have been left to fixate on this set of foundational numbers –which respectively quantify the heuristic concepts of "none", "some" and "all", along with the respective names of "zero", "one" and "infinity". Note that the alignment of "some" in symbolic logic with "one" in mathematics arises because in logical systems the term "some" denotes "1 or more", while in mathematical systems. (as will be shown in the formulation of a

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co-ordinate system in Chapter 2, Section 2) the arbitrary choice of a measuring reference is usually standardized, irrespective of what is being quantified, by focusing on some pre-selected entity being assigned a magnitude equal to 1.

Furthermore, in the historical evolution of mathematical thought, these three numbers have produced paradigm shifts in understanding "what", "how", and "why" in mathematics. An example of this, which is one of the main compasses for this book, is a treatise written in 1941 by Richard Courant and Herbert Robbins which set out to answer the question: "What is mathematics?" Such a choice was made not only because this classical treatise [3] is still one of the standards of excellence today, which has remained in wide circulation for three-quarters of a century since its original publication, but also the observation that the intended audience of that book is the one that this author hopes to reach with his treatise. Unlike these two mentoring references, which cut a wide swath of the domain of mathematics, the monograph which follows is limited to that subdivision of mathematics subsumed by the term "calculus", along with emphasizing its place in the larger picture of intellectual inquiry. The following remarks, extracted from the preface of [3], expresses precisely the audience to whom this monograph is directed:

"For more than two thousand years some familiarity with mathematics has been regarded as an indispensible part of the intellectual equipment of every cultured person. . . . Teachers, students, and the educated public demand constructive reform . . . It is possible to proceed on a straight road from the very elements to vantage points from which the substance and driving forces of modern mathematics can be surveyed. . . . The present book is an attempt in this direction. . . . It requires a certain degree of intellectual maturity and a willingness to do some thinking on one's own. This book is written for beginners and scholars, for students and teachers, for philosophers and engineers, for class rooms and libraries".

Keywords: Algebra, All, Counting, Denotation *vs*. Connotation, Foundational Numbers, GIGO and Half-right Answers, Ideas, Indeterminate Forms, Infinity, Inverse, None, Numbers Denoting, One, Primitive, Repetition, Some, Twelfth Commandment, Why *vs*. How, Word *vs*. Term, Zero.

1.1. INTRODUCTION

To say that the understanding and teaching of calculus has not changed much in over three centuries may be overkill. Then, again, it may not be! Upon reviewing all of the calculus books on the shelf in any large university library, the astute observer will become aware just how staid the associated pedagogy actually is! Not only will they conclude that, the subject matter which was developed then still loudly resonates today, but also that the focus of the respective past and present authors has always been related to HOW, in contradistinction to WHY. Notwithstanding the inclusion of computer graphics, which assists in visualizing selected applications, there has been a scarcity of new ideas to assist comprehending the foundation upon which the underlying mathematics has been built. Had the subject matter been readily understood, this would not be a liability. To the contrary, most otherwise educated persons consider the very word “calculus” as being synonymous with “way beyond comprehension”.

This treatise was formulated in order to demonstrate that, when properly presented, the fundamentals of calculus are the very same ideas that were presented in those earlier, more intuitive, and consequently deemed to be more elementary, studies. In the same manner as the recognition that the number “zero” was a useful concept when it was postulated eleven centuries ago, a similar useful number designated by the term “infinity” underlies a different, but in many ways comparable, extension of the number system. These two numbers (zero and infinity) when combined with the most primitive of all numbers, one, forms the above mentioned foundational set. Moreover, combining this three member set with the six fundamental operations of arithmetic extends the domain of arithmetic into a larger domain which is designated by the name “calculus”. The approach taken in this treatise rejects the premise that calculus is a whole new subdivision of mathematics which is predicated on an arcane set of previously developed algorithms and which is primarily mastered by memorization. To the contrary, the domain of calculus is viewed as a direct extension of the arithmetic learned in the earlier grades of grammar school and its extension into high school algebra.

At this point, attention is further directed to an important distinction between the nouns “term” *vs*. “word”: namely, “term” is reserved for a precise denotation by professionals in a given field of study in contradistinction to “word”, which is in common usage by lay persons. Moreover, one associates “precision” with how

CHAPTER 2

Re-Examination of Basic Algebra and Trigonometry

Abstract: As a prerequisite to appreciating that domain of mathematics referred to as "calculus", this chapter re-examines important ideas supposedly (or maybe one should say "hopefully"), learned in previous studies. The author's objective in including this chapter is to emphasize (and thus help to understand) WHY, in contradistinction to merely HOW, algebraic operations are performed. Notwithstanding that this set of topics had been developed in previous encounters with mathematics, they are now viewed from an advanced viewpoint. One begins by reiterating that over a millennium ago arithmetic was simplified by assigning a number (zero) to "nothing"; thereby causing a paradigm shift that brought mathematics into the mainstream. A similar new paradigm shift, focused on a number that represents the concept of "all" (in that philosophical trichotomy of none, some and all) is herewith proposed. This role will be filled by a new number, denoted as "infinity", which includes the infinitely large, the infinitely small, and the infinitely dense. Having made such an introduction, the rest of this treatise, starting with Chapter 3, examines the relationship between the set of three "foundational" numbers (zero, one, and infinity) upon which, we assert, the development of calculus should be formulated, and the familiar arithmetic operations of compounding and undoing previous operations.

Keywords: Coordinate Systems (Postulates, Cartesian, Polar), Division Involving Zero, Equations of a Straight Line (5 Common Forms), Functions, Graphing: Plotting Points *vs.* Characteristic Curves, Logarithms, Measurement, Memorized Rules: Why Selected Ones Work, Plane (Circular) Trigonometry, Spherical Trigonometry, Symmetry.

2.1. FUNCTIONS AND FUNCTIONAL EQUATIONS

A "function" in mathematics is defined [12] as: "A mathematical expression describing the relation between variables". For a given literal number, consider a related second literal number. The first of these literal numbers is denoted as "an independent variable". The second is denoted as "a dependent variable". For

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instance, select the equation $y = 2x^2 - 1$. Next, choose any value for x. For that value of x, there is a unique corresponding value for y; namely:

if $x = 0$, then $y = -1$

if $x = 2$, then $y = 7$

if $x = -10$, then $y = 199$, *etc*.

In this example, y was written alone on the left side of the equal sign. Only terms not involving y were written on the right side. Such an equation is called "an explicit function of x". The term "explicit" means that given a value for x, the value of y is explained. This is done by direct computation. It is not necessary to solve any equations to determine the value for y. Instead, one merely performs the operations in the order stated.

This same equation could also have been written as:

$y - 2x^2 = -1$

$y + 1 = 2x^2$

$2x^2 - y - 1 = 0$, *etc*.

In each of these last three equations, the value for y was implied by the equation and the given value of x; therefore y is denoted as an "implicit" function of x. Alternately, all four of these equations could be viewed as though y was the independent variable. Now x, the dependent variable, has the format of an implicit function of y; namely:

$$x = \pm\sqrt{\frac{(y+1)}{2}}$$

Although such a scenario is viable, it is not traditional. Instead the established custom is to restrict functions to being single-valued. When dependent variables can have more than one value they are called “relations”. Relations are usually sub-divided into distinct functions, rather than staying lumped together. In this example one function has a positive sign; the other a negative sign.

A further limitation that is often made is to restrict independent variables to being real numbers. Had the example chosen been $x = 2y^3 - 1$, it is important to remember that for polynomial equations having real coefficients, there are as many roots as the highest exponent of the dependent variable and that at least one of these is real. The other two roots may be either real or imaginary.

What may seem to be “simple” extensions, such as into the domain of complex numbers, may be anything but what our intuition expects. It is important to be aware that the adjective “simple” and the adverb “simply” are probably two of the least simple words that any student of mathematics will ever encounter. The knowledge that they have accumulated in the past will determine what is, and what is not, simple. For example a “simple figure” in geometry requires many attributes not anticipated by the beginning, or even many of the more advanced, students. A small set of such terms include “simply-closed”, “simply-connected”, “orientable”, “non-redundant”, *etc*. Some of these will be encountered in later chapters.

Additional “simple” examples of functions include:

(1) The area of a square: $A = s^2$. The side length s is the independent variable. The area A is the dependent variable. A is an explicit function of s. By taking the square root of both sides, s is an implicit function of A. Additionally, because of the science, rather than the mathematics, s is single-valued: Only in rare circumstances are negative lengths considered important. For nearly all examples, these solutions are extraneous.

(2) For any circle, the circumference, $\underline{C}$, divided by the diameter, $\underline{D}$, is a constant. This quotient, $\left(\frac{C}{D}\right)$ is an irrational number written as the Greek letter π. The numerical value of pi is approximately 3.14159. C is an implicit function of D, and D is an implicit function of C. In order to write this equation as an explicit function of D one needs merely to multiply both sides of the equation by D;

Calculus: The Logical Extension of Arithmetic, 2016, **109-185**

CHAPTER 3

Zero Divided by Zero − and the Concept of Differentiation

Abstract: This chapter begins with a brief historical introduction, wherein the seeming paradox of Achilles and the turtle is examined. Although this treatise does follow part of the tradition and progresses to the concepts of limits and continuity, including the more formal perspective of using epsilon and delta type proofs that have been the touchstone of calculus's foundation for over three centuries, no further development of such a protocol is undertaken. In its place a very different "Weltanschauung" (world philosophy) that focuses on what this author asserts is the appropriate underlying foundation of calculus is promulgated; namely, the relation of the concepts of none, some and all (algebraically expressed as 0, 1 and ∞) to the six fundamental operations of numbers (addition, subtraction, multiplication, division, raising to a power and extracting a root). From the 54 potential binary combinations of these sets, the seven traditional indeterminate l'Hôpital forms, as well as three additional related forms that mathematicians have missed for over three centuries are distilled. In the process, attention is focused on combinations deliberately disallowed in previous mathematics courses; especially those that arise with respect to infinity and division by zero. One particular combination, which has as its objective the determination of those extreme values that the given function can reach both globally (over all of space), and locally (in a given interval), is postulated to be the foundation upon which, provided the appropriate constraints are included, the first of the major techniques of calculus is to be built. The philosophy espoused herein views a specific related function, derived from the given function and thus named as "the derivative of that function", as the division of two, considered to be even more elementary, functions, called "differentials". Each of these differentials, which are primarily algebraic constructs, is equivalent to having a limit value of 0. Consequently, the derivative may be viewed as giving meaning to the indeterminate form $\frac{0}{0}$, under a set of constraints to be designated at a later time. Meanwhile, selected other entities, which had been historically defined, such as the concept traditionally expressed as "concavity", are viewed as having been relegated to the status of insignificance. This is, in contradistinction to many traditional

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calculus textbooks which belabor concavity as being nearly equal in importance with the extreme values of maxima and minima. The topological subtleties, often forming the basis of theoretically biased courses, are included only when they add to an intuitive understanding of the subject matter, and thus become of interest to applied scientists and engineers.

Two other l'Hôpitalian combinations, which are similarly depicted as forming the foundation for the other two significant terms that comprise the principal domain associated with calculus will be introduced and developed in Chapters 4 and 6 respectively.

Keywords: Arithmetic Operations Involving Infinity, Continuity, Curve Sketching, Derivatives (Definition, Poly- *vs*. Multi-nomials, Trig Functions), Differential Calculus, Epsilon-Delta Processes, Extrema (Maximum, Minimum, Point of Inflection), Implicit Differentiation, "Last" Number and Interpreting Infinity: l'Hôpital Indeterminate Forms *vs*. l'Hôpital-Elk, Indeterminate Forms, Limits, Related Rates.

3.1. LIMITS AND CONTINUITY

The foundations of what was to become the basis for calculus, two millennia later, was a paradox proposed by the Greek philosopher Zeno:

Suppose Achilles, the fastest human in mythology, ran a foot race with a turtle. To make the race "fair", the turtle was given a head start. Zeno now declared that, by the simple application of logic, not only could Achilles not win the race, he could NEVER catch up to the turtle. Moreover, neither of them could ever finish the race.

The supporting logic of his argument is described as follows:

Designate P_0 as the starting point for Achilles and P_1 the starting point for the turtle. Denote by t_1 the time it takes for Achilles to run from P_0 to P_1. In this time interval the turtle will have advanced to P_2.

Let t_2 be the time it takes for Achilles to run from P_1 to P_2. In this time interval the turtle has advanced to point P_3.

Let t_3 be the time it takes for Achilles to run from P_2 to P_3. In this time interval the turtle has advanced to point P_4.

This process can be continued indefinitely — an infinite number of times, with at the end of each time interval, the turtle always being ahead of Achilles.

Consequently, one can conclude that, inasmuch as an infinite number of time intervals has passed:

(1) Achilles can NEVER catch the turtle

and

(2) Neither of them can EVER finish the race.

The flow of logic is herewith illustrated as Fig. (**3.1-1**).

P_0	P_1				Finish
A	T				
position of Achilles (A) and the turtle (T) at t = 0					
P_0	P_1	P_2			Finish
	A	T			
position of Achilles (A) and the turtle (T) at t = 1					
P_0	P_1	P_2	P_3		Finish
		A	T		
position of Achilles (A) and the turtle (T) at t = 2					
P_0	P_1	P_2	P_3	P_4	Finish
			A	T	
position of Achilles (A) and the turtle (T) at t = 3					

Fig. (3.1-1). Relative progress of race: Achilles *vs*. the turtle.

CHAPTER 4

Infinity Times Zero and the Concept of Integration

Abstract: In this chapter, the second of the processes in the traditional sequence of courses that comprise the calculus curriculum, “integral calculus”, is examined in terms of its algebraic foundation. A new paradigm is proposed that views “integration”, also often designated as “anti- (or inverse) differentiation”, as a form of infinity multiplied by zero. Such a protocol is demonstrated as being the inverse operation to the previously developed process of differentiation in Chapter 3. However, unlike the function produced by the process of differentiation, this inverse is either not unique and needs to be supplemented with an arbitrary unspecified constant (which is addended to the generated function) or a set of limits. Along with “sloughing through” several of the techniques associated with such anti-differentiation, the mathematical underpinning of this inverse operation is introduced. This is then supplemented by an expansion of the horizon of “what is mathematics?” to introduce (1) a special (more advanced) function, called the Dirac delta function, which, in an altogether different manner is also subsumed by the over-arching concept of infinity multiplied by zero, and (2) the theoretical base from one limited to continuous functions (called “Riemann integration”) to a larger set that includes selected discontinuous functions (called “Lebesgue integration”).

Keywords: Anti-Derivatives, Area of Surface of Revolution, Dirac Delta, Direct Substitutions, Indirect Substitutions, Integral Calculus, Integration By Parts, Kronecker Delta, Lebesque Integral, Length of Planar Curves, Volume of Revolution (Slices *vs*. Shells).

4.1. ANTI-DERIVATIVES

In the previous chapter, the procedure used to find that special related function called the "derivative of a given function" was developed, along with the associated mathematics. This chapter now probes the inverse operation; namely,

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how to start from the answer that one got by differentiating a function and work backwards to uncover what the original function was. In many ways this process is comparable to factoring in algebra.

Just as the mechanics of factoring involves guessing two or more answers which when multiplied together produces the original term that one wanted to "simplify", for integration, one similarly guesses an answer which, when differentiated, hopefully will generate the desired term. In both scenarios, guessing the correct answer may range in difficulty from being "very straight-forward and easy" to being "far from obvious". For example, the trivial, almost intuitive, guess when factoring $x^4 - 16$ is: $(x^2 + 4) * (x^2 - 4)$, which may be further simplified to $(x^2 + 4) * (x - 2) * (x + 2)$. On the other hand, when the sign between the two terms that comprise the quantity being factored is changed from - to + (*i.e.*, $x^4 + 16$) either the factors are: $(x^2 + 2\sqrt{2}x + 4) * (x^2 - 2\sqrt{2}x + 4)$ when one is constrained to stay in the domain of real numbers, or else requires entering the domain of complex numbers. Furthermore, had the function being factored been the supposedly simpler $x^2 + 16$, the "factors" would involve the square root of the variable x. Consequently, instead of this being in the far simpler domain of polynomials (taught in primary school classes) it would now require the introduction of multinomials, which is certainly not a simplification.

In a like manner, many known integrations have been similarly discovered through a trial and error process or, one might say, by serendipity. In other words, integration, like factoring, is a process with no known universally applicable algorithm. Nevertheless, once that unique term or combination of terms has been shown to have been retrieved by a single direct differentiating, the starting function along with its now known integral is entered into a “library” (of known integrals) for future usage. In other words, one may readily and correctly conclude that integration (or as it is often called anti-differentiation) is an "art" - mastered only by practice, rather than a “science”, wherein the etymology of the word “science”, comes from the Latin stem “scio”, “I know”.

The process of developing this art begins by noting that for the general polynomial function:

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$$

the derivative is:

$$y' = a_1 + 2a_2x + 3a_3x^2 + ...$$

In other words for each of the individual terms of the given function, the derivative has as its exponent a numerical value of one less than the starting exponent and that the given coefficient has been multiplied by this original value of the exponent. Consistence is now achieved by recognizing that the given constant term (a_0) may be viewed as though it had been written in its equivalent form a_0x^0, since $x^0 = 1$. Consequently, the derivative of a constant is equal to 0.

From the above paragraph, one concludes that each term of the anti- derivative of a polynomial will have as its exponent exactly one more than the exponent of the starting function and, as its coefficient, the original term divided by the original exponent. For example, had the term in the given polynomial function been $5x^3$,

CHAPTER 5

Analytic Geometry

Abstract: In the now traditional subdivision of mathematics into algebra, geometry and analysis popularized by Felix Klein over a century ago, calculus was the first subset of analysis taught in the educational system. Meanwhile, precisely because many of the uses of calculus were associated with a geometrical perspective, a special subset of mathematics, called "analytic geometry", was formulated and taught (sometimes interdigited and sometimes as a separate course) concurrent with the underlying foundations of calculus. In particular, as well as the basic algebra discussed up to this point, an entire chapter is herewith interjected before continuing on this author's path to understanding "what is calculus". Because one specific geometric figure, the hyperbola, will be shown to play an important role in that third fundamental level of complexity, exponentiation and root extraction, this treatise diverts its focus from the l'Hôpitalian aspects associated with extending algebra beyond the real finite domain in order to probe what may be considered to be a "tangential" path. (Here the word "tangential" has the layman's connotation of moving away from a central idea that is in the process of being developed. This is in contradistinction to any role as a term in trigonometry).

In an attempt to remove the blinders that have been firmly fastened onto the student by the traditional presentation, this treatise eschews always remaining focused on a single dimension (be it either a line, a plane or a three dimensional embedding space) and assumes the perspective of multi-dimensionality. Additionally, even though most of the presentation is in the domain of real numbers, when appropriate, the place of complex numbers in the overall scheme of "analysis" is not overlooked. This will be especially useful in understanding the correlation between traditional (circular) trigonometry of Chapter 2 and a differently focused "hyperbolic" trigonometry, which will be the focus of Chapter 6.

Keywords: Analytic Geometry, Conic Sections, Cylinder, Ellipse, Ellipsoid, Hyperbola, Hyperboloid of One Sheet, Hyperboloid of Two Sheets, Imaginary Ellipsoid, Linear Equations, Parabola, Paraboloid, Quadric Cone, Quadratic

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Equations, Rotation of Axes, Translation of Axes.

5.1. INTRODUCTION TO THE ANALYTIC APPROACH

The subject matter referred to as "analytic geometry" involves setting up a coordinate system and then describing "figures" in terms of equations (or inequalities) using numbers. By such mathematical concepts, one generalizes from a specific example to a large set of "similar" figures, where, as indicated earlier, the word "similar" is a "heuristic"; *i.e.*, it lacks the precision one would hope to have in both mathematics and science. Remember that the mental picture described by the use of words is intended to be readily recognized by both the transmitter and the receiver. For example, in Chapter 4 an important distinction that exists in topology, but is often overlooked in less mathematically precise areas, is the difference between a "circle" ─ which denotes a "boundary-defined" figure whose geometric description is that of a single line enclosing an area of a plane and whose analytic descriptor is the equation $x^2 + y^2 = r^2$. This is in contradistinction to the "content-defined" segment of space designated as a "disk" ─ which denotes the set of points contained within such a boundary, and which is represented by the inequality: $x^2 + y^2 \leq r^2$.

In other words, as stressed earlier, were one to demand correctness of semantics, the area of a circle is <u>NOT</u> given by the formula $A = \pi r^2$ precisely because the term "area" implies two-dimensionality, whereas a "circle" is a one-dimensional figure. Consequently, the technically correct statement would be: *the "area" of any "circle" is zero.* This is notwithstanding that most persons, including most mathematicians, would view such a statement as pedantry. This is notwithstanding that figures having a specific heuristic of shape are going to be characterized algebraically using equations.

One of the simplest of these equations, which was introduced in Section 2.2., was referred to as both a first order, and also as a linear, equation:

$$Ax + By + C = 0, \qquad \textbf{(1)}$$

Now, the focus is directed to the next higher domain wherein either one or both variables in a single equation involve only integer second order or lower terms. Such equations are called "quadratic". For such equations there are two common approaches:

(i) The simpler of these is from the functional perspective. Here, only one of the variables is viewed as being independent. Such a "quadratic equation in one variable" is traditionally written explicitly as

$$y = ax^2 + bx + c \qquad \textbf{(2)}$$

This equation, as demonstrated in algebra courses, has its solution when y is set equal to 0:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textbf{(3)}$$

Notwithstanding that, from an intuitive perspective, one might proceed to higher (namely third order, denoted as "cubic", fourth order, denoted as "quartic", *etc.*) order equations in this single variable, **x,** and the functional relationship y = f(x) which is then "solved" by having this second variable, y, set equal to 0, the

Chapter 6

One Raised to the Infinite Power, the Concept of Natural Logarithms, and Hyperbolic Trigonometry

Abstract: This chapter returns the overall focus of this treatise to the development of those unique attributes of calculus begun in Chapters 3 and 4; namely differentiation and integration. Having diverted attention in order to follow what was a seemingly tangential path of analytic geometry in Chapter 5, the main stream of calculus is returned to with a third indeterminate combination manifesting some unique attributes of major scientific significance. Much as the number pi was discovered four millennia ago, another constant irrational number, discovered much more recently, is associated with that particular integral for which the formula for integration of polynomials could not be extended; namely, when the exponent n had the value of -1.

The physically important constant number discovered by John Napier and designated as e in honor of Leonhart Euler [23], along with its associated function e^x, are next examined. Instead of the traditional definitions given in modern calculus textbooks, such as [14] through [21], this treatise proposes as the source of definition that binary indeterminate form 1^∞. This constant will be associated with properties common to logarithms in algebra. For such a development one reiterates the denotation and connotation of the terms “exponent” and “logarithm”; two terms which are essentially synonymous in denotation but whose connotations are that “exponent” is usually associated with an integer or a common fraction, while “logarithm” is traditionally a decimal. In Chapter 2, the properties of logarithms were developed for any base, but primarily when the base of the numbering system was the number 10. Such logarithms were designated with the adjective “common” in conformity with the basis of our number system being the biological “accident” that the human species has evolved to have ten fingers.

Meanwhile, a similar system, called “natural logarithms”, in which a different base number designated as e, will be shown to have mathematical importance. This new variety of logarithm and this special number e are encountered in diverse fields such as science and economics.

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The definition of e will be in terms of a definite integral with the argument of the function being one, or both, of its limits and the integrand being a “dummy variable”. The added perspective that will accompany this fundamental constant of both nature and of advanced mathematics, e, will be a third member of the set of indeterminate forms, cataloged by l'Hôpital in 1696, which was discussed, but not then assigned a name other than L_7 in Section 3.2; namely one raised to the infinite power (1^∞).

Next, employing algebraic properties associated with exponents, a pragmatic technique, which is applicable to functions that are combinations of multiplication, division, exponents and roots, while simultaneously being limited with respect to addition and subtraction will be described. This technique, called logarithmic differentiation, has been devised so as to decrease the tedium of selected traditional differentiations in many instances by employing properties of algebra.

In a similar manner, this number e, as the base of the function e^x, has the important functional identity property that its derivative is equal to the function itself. Moreover, every higher derivative (and integral, when the constants of integration are set equal to zero) is also equal to this function. It is now further observed that the sum and difference of this function and its reciprocal bear a seemingly serendipitous relation to the respective cosine and sine functions of trigonometry. It is precisely one-half of each of these two relations which have been the traditional nearly-universal definition of that group of functions referred to as the hyperbolic trigonometric functions; i.e., $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$. To the contrary, this treatise defines such a set of functions, as their names imply, starting from the geometry of a selected reference hyperbola. By the proposed geometric definitions, all of the identity properties of these functions are derivable without reference to their relation to exponential functions. In other word, the exponential relationships are downgraded to being secondary properties, while the trigonometry of the reference hyperbola is elevated to being the basis for definition. That the exponential relations are, in fact, valid is an intrinsic property of the confluence of algebra and geometry. This will be shown in Chapter 7 by expanding each of these functions using infinite series. Consequently, each is a different path to the same mathematical description in much the same manner as the set of six fundamental functions in circular trigonometry was defined by starting from either angles in a right triangle or lengths in a unit circle.

Keywords: Exponential Function e^x, Hyperbolic Trigonometry, Derivation from

Hyperbola *vs*. Exponential Functions, Logarithmic Differentiation, Naperian Constant e, Natural Logarithms.

6.1. DEFINING THE "NATURAL" LOGARITHM AND THE NAPERIAN BASE

Up until this time, all of the functions of interest had been of the form f(x) = some algebraic or trigonometric combination of multinomial. The domain of functions may now be expanded to include integrals as a part, or all, of the variable. As well as the more familiar material in earlier chapters, one may define a function f(x) in which the unknown x occurs only in the limits of an integration. For such a scenario the function being integrated is independent of what name is associated to that variable (which is assigned the name "dummy variable"); *i.e.*, the name chosen for this variable is immaterial. One could just as well call the variable **y** or **α** or anything else which represented a "number"; *e.g.*, $f(x) = \int_a^x f'(y)dy$ or $\int_a^x f'(\alpha)d\alpha$, *etc*., where **a** is a known constant and f '(x) is the derivative of f(x). Alternately, one could have other combinations, such as $f(x) = \int_x^{2x} f'(y)dy$, *etc*.

In particular, begin by selecting as the integrand that one value from the polynomial function that was unable to be integrated in Chapter 4; namely n = −1; *i.e.*, $f(x) = \frac{1}{x}$. The graph of this function, which is a rectangular hyperbola with branches in the first and third quadrants, is illustrated in Fig. (**6.1.1**). At x = 0, not only is the value of this function infinite, it is intrinsically discontinuous with $f(0^-) = -\infty$ while $f(0^+) = +\infty$. Next, one observes the area between the curve and

CHAPTER 7

Infinite Sequences and Series

Abstract: Up to this point in the proposed new perspective for understanding "what is calculus?" the domain associated with both integration and differentiation has mostly been confined to continuous functions. As a concluding chapter of this opus, the focus is directed to a discussion of discrete variables with an examination of the domain of sequences and series; then a re-definition of important functions, in particular trigonometric and exponential functions, in term of infinite series, and a broad look at the concept of infinity as both a cardinal and an ordinal number.

This chapter begins by defining the concept of sequences and both the mathematical limitations and the heuristic expectations that are fundamental to a quantitative, as well as a qualitative, development of the question "is the sequence of counting numbers unending?" and the related question "if there is such a "last" number, to which the name "infinity" has been given, what are its properties?" In the preceding chapters one observed that infinite concepts applied not only to being "infinitely large", but also to being "infinitely small". To this latter category the term "infinitesimal" was applied. In this chapter, the further concept, referred to as different "orders" of infinity, will be encountered. Emphasis will be placed on a concept that this author prefers to associate with the heuristic of being "infinitely dense", in contradistinction to one of being "infinitely large".

7.1. INTRODUCTION TO SEQUENCES

After a superficial introduction in Section 1.1, the attention of this treatise has been focused almost exclusively on functions in a space which have the property that between any two points there is always another point. This has led to development in that mathematical field which has the heuristic concept of "continuity", along with the further concept of differentiability. In this chapter the initial focus will be on a domain in which the functions are discrete, rather than continuous. For such a field a different concept of infinity will be involved.

The term "sequence" denotes a collection of items assigned successively numbered names: $\{a_1, a_2, a_3, \ldots\}$. Any single letter (in any recognized language) with a numerical subscript, such as a_n, denotes the n-th "member" of that sequence. Subscripts usually, BUT NOT NECESSARILY, start with 1.

An interesting comment in science concerns the extension of a given class of moieties for which an important sequence, starting with "1" has been categorized. This sequence has then been recognized as a subdivision of a larger, more encompassing, taxonomical class of less closely bonded moieties with the connecting entity being designated as the precursor, or "zeroth" member of this larger set. In chemistry, such a protocol underlies the basis of chemical taxonomy and nomenclature [4]. For such a set the subdivision into "functional groups" in organic chemistry provides many examples: Although most congener sequences start with n = 1, there is often a useful meaning, which includes a far larger set when n = 0. For example, the empirical formula for alkanes is: C_nH_{2n+2}, where the structural formula is: $H - (CH_2)_n - H$. Such a sequence may be extended to include n equals 0; *vis*., C_0H_2, which is just another name for H_2; *i.e.*, a hydrogen molecule. The non-repeating; *i.e.*, "zeroth member" end modules are two hydrogen atoms, which when joined together form a single hydrogen molecule. Similarly, many other large classes of molecules are formed by dissecting a double bond and connecting successive modules into a "one dimensional" chain. For example, polyethylenes are often viewed as starting from an ethylene module and then joining that module with a second such module, *etc*.

There exist many important sequences in mathematics and science. The set of prime numbers given in Table **7.1.1** fulfills all of one's heuristic expectations.

Table 7.1. 1. The sequence of prime numbers.

N	1	2	3	4	5	6	7	…
P_N	2	3	5	7	11	13	17	…

Two important terms associated with a given sequence of numbers are "cardinal number" (how many elements there are in this sequence) and "ordinal number" (where in this sequence a pre-selected cardinal number is located). For example, the cardinal number 7 is the fourth prime number. Because most of the interesting mathematics associated with sequences occurs when that sequence has an infinite number of terms, it is often desired to determine what cardinal number is associated with infinity. For the case of prime numbers, one of these heuristic expectations, going back to Euclid in about 300 B.C., is the hypothesis, and his readily demonstrated "proof", that there are an infinite number of prime numbers. Euclid's proof was to assume the contrary and then illustrate that such an assumption is contradicted by logical thinking. The problem with this proof, as well as all comparable indirect proofs, is the arrogance to assume that the formulator has the ability to include all possibilities. Each historical description of the number system (from "natural numbers", through "integers", through "rational numbers", through "real numbers", through "complex numbers") has assumed it had included all possibilities, only for a future mathematician to find first an "outlier" and then to have it recognized as an example of a still larger set of numbers. This treatise is proposing that the concept of infinity not be considered as a unique number, but rather let it have a wider domain, as will be described below.

Our attention in this treatise is now focused on various sets in science and mathematics that may, or may not, be regarded as sequences. For eighty-six years the set of planets in our solar system $\{P_N\}$: Mercury - Venus - Earth - Mars - Jupiter - Saturn - Uranus - Neptune - Pluto was accepted as "accurate" through these nine "terms". In 2006, however, this sequence was reclassified into being either only the first eight, or else expanded to a set of twelve, planetary members, depending on which attributes were to be included in the denotation of the term "planet". The status of Pluto was relegated to being just another one of thousands of Kuiper Belt objects, rather than its former classification as a planet. In its stead three, what are now called "dwarf-planets" (Makemake, Haumea and Eris) plus Pluto were assigned to a new related set, referred to as "planetoids"). Note that the set of Kuiper Belt objects does not lend itself to a unique ordering, based on any mathematical principle such as distance of its members from the sun; consequently, it does not meet the requirement to be categorized as a sequence.

SUBJECT INDEX

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Seymour B. Elk

Seymour B. Elk is a mathematical chemist with a new, hopefully paradigm-shifting, perspective on the place of calculus in the evolution of mathematical thought. His educational background, being a Doctor of Science degree, included studies in mathematics, physics and chemistry from Yale University, Purdue University, Washington University of St. Louis, University of Zurich and Eurotechnical Research University. He began the first decade of his professional career as a mathematician-scientist-engineer-administrator in the American aerospace industry.

He was then appointed as the Chairman of the Department of Mathematics and Science at Parks College of Aeronautical Technology of Saint Louis University. This was the beginning of a second career in academia and research, culminating in a full professorship with a joint appointment in mathematics and chemistry at the University of Bridgeport. However, that university came under the supervision of Reverend Sun Myong Moon's Unification Church; he then pursued his research goals, as well as entrepreneurial activities. During this time, he authored two innovative research monographs "A New Unifying Biparametric Nomenclature that Spans all of Chemistry" (Elsevier 2004) and "The Structure-Nomenclature Cycle of Chemistry" (Mathematical Chemistry Monographs #11, University of Kragujevic, Serbia 2011) as well as over five dozen research articles in leading American and European mathematics and chemistry journals, promoting chemical structure as a branch of applied geometry/topology and of number theory.

Additionally, contrary to the traditional emergence of calculus, as formulated by Leibnitz and Newton, a third of a millennium ago, that has monopolized the development of mathematical analysis ever since, this treatise regards the subject matter subsumed by the term "calculus" as merely the logical extension of augmenting elementary arithmetic with the recognition that there is no last number in the counting scheme. When such a world view is then combined with the binary arithmetic of the six elementary operations of arithmetic (addition, subtraction, multiplication, division, exponentiation and root extraction) and the 54 possible binary combinations of the foundational numbers (zero, one and infinity), one is then able to distill the calculus operations of differentiation and integration, as well as the Naperian base number e.

Among other major innovations included in this monograph are: (1) the development of hyperbolic trigonometry from the corresponding relations in circular trigonometry, rather than the ubiquitous historical perspective that has defined the hyperbolic trig functions in terms of serendipitous combinations of exponential functions, and (2) a superficial introduction to the relation of calculus to other aspects of advanced mathematics, such as the Dirac delta function, Lebesgue integration, the elementary geometry associated with contravariance vs. covariance in tensor analysis, etc.